

Long-range topological insulators and weakened bulk-boundary correspondence

L. Lepori^{1,2,*} and L. Dell'Anna²

¹*Dipartimento di Scienze Fisiche e Chimiche, Università dell'Aquila, via Vetoio, I-67010 Coppito-L'Aquila, Italy.*

²*Dipartimento di Fisica e Astronomia, Università di Padova, Via Marzolo 8, I-35131 Padova, Italy.*

We formalize the appearance of new types of insulators in long-range fermionic quantum systems. These phases are not included in the famous "ten-fold way classification" for the short-range topological insulators. This conclusion is illustrated studying at first specific one-dimensional long-range examples, in particular their phase diagrams and contents in symmetries and entanglement. The purely long-range phases are signaled by the violation of the area-law for the Von Neumann entropy and by corresponding peculiar distributions for the entanglement spectrum. Later on, the origin of the deviations from the ten-fold way classification is analyzed from a more general point of view. In particular, it is found related with a particular type of divergences occurring in the spectrum, due to the long-range couplings. A satisfying characterization for the purely long-range phases can be achieved, at least for one-dimensional quantum systems, as well as the connected definition of a nontrivial topology, provided a careful evaluation of the long-range contributions. Our results induce to reconsider carefully the definition of correlation length in long-range systems. The same analysis also allows to infer, at least for one-dimensional models, the weakening of the bulk-boundary correspondence, due to the important correlations between bulk and edges, and consequently to clarify the nature of the massive edge states appearing in the topological long-range phases. The emergence of this peculiar edge structure is signaled by the bulk entanglement spectrum. The stability of the long-range phases against finite-size effects, very relevant in current experiments, and against local disorder is also discussed, showing notably that the latter ingredient can even strengthen the effect of the long-range couplings. Finally, we analyze the entanglement content of the paradigmatic long-range Ising spin chain, inferring again important deviations from the short-range regime, as well as the limitations of bulk-boundary (tensor-network based) approaches to classify long-range spin models.

I. INTRODUCTION

The study of topological phases of matter experienced a growing interest in the last decades. In the absence of interaction, a central result is the complete classification of the topologically inequivalent (families of) phases for fermionic systems, the famous "ten-fold way classification" [1–5]. The systems included in this scheme host a "symmetry-protected topological order", indeed their nontrivial topology is constrained and protected by some discrete symmetries, oppositely to genuine topological order. This theoretical achievement have been confirmed and corroborated by the experimental characterization of solid-state compounds with topological properties [6–9].

In spite of an energy gap obstructing in general charge or spin bulk conductivity, the main macroscopic property exhibited by aq nontrivial topological insulators is the presence of edge conductivity, due to massless modes localized therein and well distinct from bulk excitations. Moreover, phases with different topology are separated each others by continuous transitions, where the bulk mass gap vanishes. Concerning the entanglement properties, the matter included in the ten-fold way classification displays short-range entanglement and correlations, the opposite situation holding again for genuine topological order [10].

All the mentioned results are specific for quantum systems described by Hamiltonians with short-range terms only. However, in the last years also the study of long-range classical and quantum systems [11], both at and out of the equilibrium, gained a renewed attention.

Independent theoretical studies have shown that long-range quantum systems can exhibit various peculiar features, mostly stemming from the breakdown of lattice locality [12–26]. This set includes static correlation functions with hybrid (exponential and algebraic) decay [27–30], anomalous growth for the entanglement after quenches [31], new constraints on thermalization [32].

Even more interestingly, very recent works [25, 27, 28, 33–39] have suggested that long-range systems can host new phases at sufficiently small values of the decay exponents α for the Hamiltonian terms. These phases often manifest interesting features not owned by the short-range ones, including continuous quantum phase transitions without mass gap closure, violation of the area-law for the Von Neumann entropy and of the Mermin-Wagner theorem, emergence of massive edge modes.

The occurrence of these properties, some of them also checked in experiments of trapped ions [40, 41], opened various issues and problems. In [27, 35, 36] it has been inferred that, for not interacting long-range lattice models, most of the described peculiarities can be related with the action of some states in the bulk spectrum, called "singular states", encoding some divergences related with the algebraic decay of the long-range couplings.

In spite of these important clues, the understanding of the physical origin of the mentioned purely long-range phases, as well as of their bulk and edge features, is still an open problem. Closely related, it appears a central issue to classify these phases, understanding how the ten-fold way classification evolves in the presence of long-range Hamiltonian terms, when also correlation functions have been found not exponentially decaying any longer.

In the present paper we start to investigate this problem.

* correspondence at: llepori81@gmail.com

Using first specific one-dimensional free fermionic examples and later on performing a more general formal discussion (not limited to one-dimensional cases), we show that long-range insulating phases can emerge, in some cases hosting massive edge modes, when the bulk spectrum manifests a *particular* sub-set of the mentioned singularities. The appearance of the latter singularities parallels the area-law violation for the Von Neumann entropy, a peculiar entanglement spectrum, and the related effective divergence for the correlation length, still in the presence of a nonvanishing bulk mass gap. Moreover, due to them, the definition of topology must be reconsidered *ab initio*, requiring a proper generalization of the approaches valid in the short-range limit.

Finally, we infer, for one-dimensional systems, that the so-called bulk-boundary correspondence, typical of the short-range topological insulators, gets weakened in the long-range topological phases, as well as the definition itself of localized edge state valid in the short-range limit, due to the strong long-range correlations between the edges and with the bulk dynamics.

Notably, some of the ideas and the of results achieved for one-dimensional long-range quadratic systems can hold, under specific restrictions, for higher-dimensional ones, as well as for interacting and/or spin long-range models.

The paper is organized as follows. In Section II we recall two specific examples of one-dimensional fermionic long-range quantum systems, discussing their phase diagrams and ground state properties. In Section III we explain, giving different arguments and motivations, also from previous works, that some insulating phases hosted by these systems do not insert in the classification for the short-range topological insulators (briefly recalled in the Appendix A), but display a purely long-range nature. This thesis is reinforced in Section IV by the analysis of the entanglement spectrum for the ground states after a spatial bipartition. The same analysis is probably the main result of the manuscript, as well as the discussion of its consequences, performed in Section VII. In Section V we formalize the generic failure of the mentioned classification when long-range Hamiltonian terms are added, reconsidering the winding of maps and nonlinear σ -model approaches to it. Notably, this discussion is again not limited to one-dimensional systems. In Section VI we deal with the classification, by Berry phase and winding numbers, of the long-range phases encountered in the previous Sections, as well the limitations and open problems concerning these approaches. At then end, we address the generalization of these methods to long-range free fermionic models with different symmetries and dimensionality. In Section VII we analyze at first the definition and the behaviour of the correlation length in long-range systems. Later on, starting from the latter discussion and from the results on the entanglement spectrum, we infer the weakening of the bulk-boundary correspondence in the long-range topological phases, clarifying the nature of their massive edge modes. In Section VIII we discuss the stability of the long-range phases against finite-size effects and local disorder, expected to smear the effects of the long-range Hamiltonian terms. In Section IX we probe the possible extension of some results obtained so far to other long-range mod-

els, spin-based and/or interacting. For this task, we discuss the paradigmatic case of the long-range Ising model, finding again peculiarities in the entanglement content at small enough α . Conclusions are finally presented in Section X. Further details, mentioned in the main text but not immediately required to understand it, are given in the Appendices A-E.

II. LONG-RANGE KITAEV CHAINS

We discuss in this Section two long-range (LR) generalization of the short-range (SR) Kitaev Hamiltonian [42]. We closely follow the references [27, 28, 35]. Further details are given in the Appendix B.

A. The models

In [27, 35, 36] two quadratic quantum models involving spinless fermions on a one-dimensional lattice have been studied extensively. The first one is characterized by a LR pairing:

$$H_{\text{lat}} = -w \sum_{j=1}^L \left(a_j^\dagger a_{j+1} + \text{h.c.} \right) - \mu \sum_{j=1}^L \left(n_j - \frac{1}{2} \right) + \frac{\Delta}{2} \sum_{j=1}^L \sum_{\ell=1}^{L-1} d_\ell^{-\alpha} \left(a_j a_{j+\ell} + a_{j+\ell}^\dagger a_j^\dagger \right). \quad (1)$$

For a closed chain, we define in Eq. (1) $d_\ell = \ell$ ($d_\ell = L - \ell$) if $\ell < L/2$ ($\ell > L/2$) and we choose anti-periodic boundary conditions [27].

The spectrum $\lambda(k)$ of the Hamiltonian in Eq. (1) displays a critical line at $\mu = 1$ for every α and a the critical semi-line $\mu = -1$ for $\alpha > 1$. For more details see the Appendix B. Notably the energy of the quasiparticles diverges in $k = \pi$ if $\alpha \leq 1$, while it displays, at every finite α and at the same momentum, divergences in some k -derivatives for $\lambda(k)$ ([27, 35]). For these reasons the states close to $k = \pi$ are called "singular states" (and their dynamics as "singular dynamics") [36]; as mentioned in the Introduction they have shown responsible of the deviations from the SR behaviours, concerning for instance the decay of the static correlation functions, the breakdown of conformal symmetry at criticality and the underlying violation of the lattice locality. Importantly, the ground state energy is still extensive also at $\alpha < 1$, in spite of the singular states, so that no Kac rescaling is required [11, 27].

For future purposes, it is convenient to report the tight-binding matrix Hamiltonian corresponding to Eq. (1):

$$H(k) = \begin{pmatrix} -(w \cos k - \frac{\mu}{2}) & \frac{\Delta}{2} f_\alpha(k + \pi) \\ \frac{\Delta}{2} f_\alpha(k + \pi) & (w \cos k - \frac{\mu}{2}) \end{pmatrix} \quad (2)$$

in the (momentum diagonal) space (a_k, a_{-k}^\dagger) . The function $f_\alpha(k) \equiv \sum_{l=1}^{L-1} \sin(kl)/d_l^\alpha$ encodes the mentioned singularities from the LR character of the Hamiltonian.

The Hamiltonian in Eqs. (1) and (2) shares the same symmetries of the SR Kitaev chain, that means, beyond the unitary Z_2 parity of the total fermionic number, the anti-unitary charge conjugation and the time reversal symmetries. This content in symmetries formally locates the model in Eq. (1) still in the class BDI of the TWC. Indeed it also holds $U_C = \sigma_y$ and $U_T = \mathbf{I}$, so that $U_C^2 = U_T^2 = \mathbf{I}$.

Some generalizations of the Hamiltonian in Eq. (1), involving as well a LR hopping, can be also considered:

$$H_{\text{lat}} = -w \sum_{j=1}^L \sum_{\ell=1}^{L-1} d_{\ell}^{-\beta} \left(a_j^{\dagger} a_{j+\ell} + \text{h.c.} \right) - \mu \sum_{j=1}^L \left(n_j - \frac{1}{2} \right) + \frac{\Delta}{2} \sum_{j=1}^L \sum_{\ell=1}^{L-1} d_{\ell}^{-\alpha} \left(a_j a_{j+\ell} + a_{j+\ell}^{\dagger} a_j^{\dagger} \right). \quad (3)$$

These models have been studied in [28] ($\beta = \alpha$). The structure and the expression for the energy of the ground-states is very similar to the ones for the Hamiltonian in Eq. (1), with the difference in Eq. (2)

$$\cos k \rightarrow g_{\alpha}(k) \equiv \sum_{\ell=1}^{L-1} \cos(k\ell) / d_{\ell}^{\alpha}. \quad (4)$$

Still divergences in the quasiparticle spectrum occur at $k = \pi$ if $\alpha < 1$ and also high-order ones at every finite α . However these divergences display a central difference compared with the ones from $f_{\alpha}(k)$ in Eq. (2): indeed $g_{\alpha}(-k) = g_{\alpha}(k)$, while $f_{\alpha}(-k) = -f_{\alpha}(k)$, affecting differently the singular dynamics, as we will see in detail in Section V. Again, the Hamiltonian in Eq. (3) shares the same symmetries of the SR Kitaev chain, then it belongs to the BDI class of the TWC.

B. The phase diagrams

Concerning the phase diagram of the Hamiltonian in Eq. (1), in [27] it has been found that above the line $\alpha = 1$ two phases take place (at $|\mu| < 1$ and $|\mu| > 1$ respectively), continuously connected with the ordered and disordered phase of the SR Kitaev chain. There the area-law Von Neumann entropy after a bipartition is fulfilled, as common in SR systems [43] (although exceptions are known in peculiar ad hoc constructed models, see e.g. [44]). This means in formulae that

$$S(l \rightarrow \infty) \rightarrow a, \quad (5)$$

where l characterizes the two parts of the chain with length l and $L - l$ (it also holds $L \rightarrow \infty$) and a is a constant. At variance, below the line $\alpha = 1$ two phases appear (at $\mu < 1$ and $\mu > 1$), signaled by a deviation from the area-law for the Von Neumann entropy. In particular this deviation turns out to be ruled by a logarithmic scaling law, analogous to the one known to hold for SR quantum systems at criticality, the so called Calabrese-Cardy scaling law [45, 46]:

$$S(l) = c_{\text{eff}} \ln l. \quad (6)$$

In Eq. (6) a value $c_{\text{eff}} \neq 0$ signals the area-law violation. The same violation has been demonstrated in [35] by an effective theory close to the critical (semi-)lines $\mu = \pm 1$.

The described phase diagram is recalled in Fig. 9 of Appendix B, where the critical semi-lines are reported, as well as the violation of the area-law.

A further striking feature of the zone below $\alpha = 1$ is that at $\mu < 1$ massive states localized on the edges appear [27], remnant of the Majorana (massless) edge modes present if $|\mu| < 1$ and $\alpha > 1$ (in the SR limit they are proper of the ordered phase for the SR Kitaev chain [42], for a review see also [47]). Indeed in [28] and hybridization mechanism of the Majorana modes, yielding the massive edge ones, has been conjectured, similar to the one occurring at finite sizes in the SR limit [42]. At every finite $\alpha > 1$, the decay for the wavefunctions of the Majorana modes displays an hybrid (exponential plus algebraic) behaviour, similar to the one of the two points static correlations. This behaviour, determining a important overlap of the edge wavefunctions, suggested the hybridization mechanism.

Interestingly, the phases at $\alpha < 1$ look continuously connected with the ones above $\alpha = 1$. In particular the phase with massive edge states at $\mu < 1$ is not separated by any mass gap closure from the ones at $\alpha > 1$, where edge modes are massless if $|\mu| < 1$ and not present at all if $|\mu| > 1$.

In spite of this fact, the real appearance of LR phases, not continuously connected with the SR ones, can be inferred from the behaviour of the Von Neumann entropy or, for an open chain, by considerations on the ground-state structure as α varies. Indeed at $\alpha > 1$ the vanishing mass of the Majorana edge modes at $|\mu| < 1$ implies the existence of two degenerate ground-states with different Z_2 fermionic parity (see Appendix A), the nonzero edge mass at $\alpha < 1$ reflects in a unique ground state with even parity. The described ground state structure deeply affects also the entanglement (spectrum) content. Indeed, from the analysis of the latter quantity, in the Section IV, another indication for the appearance of LR phases will be obtained.

Finally, the same LR phases can be detected by the approaches in Section VI, and by two indicators analyzed in Appendix B. The latter quantities exploit on one hand the superconductive nature of the Hamiltonian in Eq. (1) (a tool expected valuable also on other LR superconductive systems, say with higher dimensionality), and on the other hand the mapping of the same Hamiltonian onto a LR spin model. Notably, the phase diagram predicted in both of the cases coincides, with very good approximation, with the one depicted by the area-law violation for the Von Neumann entropy.

Without entering in a deeper detail, we mention that a qualitatively equal phase diagram has been found for the Hamiltonian in Eq. (3) with $\beta = \alpha$. In particular, at $\alpha \lesssim 1$ again an extended region where massive edge modes appear [28].

III. EVIDENCES OF DEVIATIONS FROM TWC

The phases at $\alpha < 1$ for the fermionic Hamiltonian in Eq. (1) cannot be included in the TWC for the SR topological insulators (a brief tutorial on them is given in the Appendix A). In favour of this thesis, also suspected from the results in [35] and in [37], we mention some evidences:

i) the appearance of massive edge modes in itself already signals a breakdown of the TWC, where only massless edge modes are expected. The origin of these modes will be clarified in Section VII.

ii) the TWC does not consider continuous phase transitions (e. g. between different topologies) without mass gap closure, as in the present case approaching the line $\alpha = 1$. The possibility of a first-order phase transition seems ruled out by the absence of divergences in the first derivative in α of the extensive ground-state energy $e_0(\alpha, L)$ (no cusps at every finite L). Similarly, a crossover seems excluded by the considerations about the ground-state structure outlined at the end of the Section II B. On the contrary, the absence of a mass gap closure can be justified heuristically by the large algebraically decaying tails of the static correlation functions at small values of α , also in the presence of a nonvanishing mass gap [27, 28, 35, 48]. For more details see Section VIII.

iii) the TWC involves SR (exponentially decaying) correlations and the fulfillment of the area-law for the Von Neumann entropy between disconnected subsystems. However, for the Hamiltonian in Eq. (1) below $\alpha = 1$, where the quasiparticle energy diverges, this law is violated, even if only logarithmically.

iv) the winding numbers w , characterizing the phases of the Hamiltonian in Eq. (1) at every α at least above 1, can be calculated directly following [49]. The results are correctly $w = 0$ and $w = 1$ at $\alpha > 1$. On the contrary, at $\mu < 1$ a fake semi-integer winding number $w = \frac{1}{2}$ appears below $\alpha = 1$, while $w = -\frac{1}{2}$ is obtained at $\mu > 1$ [37]. In the latter phase no edge states if found, and both the phases are characterized by a unique ground-state. These results signal a clear inconsistency in the definition of topology by the winding numbers valid in the SR limit, since these numbers, when properly defined, can assume only integer values [50, 51]. However, the mere emergence of this inconsistency can be interpreted as a signal of TWC deviation, related with the other LR features described above, as we will detail in Sections V and VI.

Finally, a bit more subtle but very relevant argument is given in [52].

A similar analysis can be performed for the Hamiltonians in Eq. (3), having the same symmetries of Eq. (1). This analysis leads to qualitatively equal conclusions as the ones just above. Indeed in [28] for the case $\beta = \alpha$ at $\alpha \lesssim 1$ again an extended region has been found in the phase diagram

where massive edge modes appear. Again this phase has a single ground-state, paralleling the nonvanishing mass of the edge states; however the same phase is not continuously connected with the disordered phase of the SR Kitaev chain.

Summing up, the arguments above yield a quite compelling evidence that the phases at $\alpha < 1$ on the Hamiltonians in Eqs. (1) and (3) escape the TWC for SR fermionic systems.

IV. FURTHER EVIDENCES FROM ENTANGLEMENT SPECTRUM

The violation of the area-law for the Von Neumann entropy at $\alpha < 1$, suggesting the appearance of new purely LR phases, induces a deeper study of the entanglement content in this gapped regime. For this reason, in the present Section we analyze the entanglement spectrum (ES) for the LR paired Kitaev chain in Eq. (1). This study will help us to determine in deeper detail the structure of the phases at $\alpha < 1$, corroborating their purely LR nature and linking each others crucial properties of them.

The ES is defined in general as the set of (Schmidt)-eigenvalues of the reduced density matrix ρ_B of a part B of the considered quantum system after a bipartition (see e.g. [53]). It is known that ES is encoding even more information than the Von Neumann entropy [54–57] and it can be calculated following the techniques described in [53].

We assume in particular a open chain with total length L and to bipartite it such to isolate a segment, say between $L/4$ and $3L/4$.

We find that, below $\alpha = 1$, in the correspondence with the violation of area-law for the Von Neumann entropy, ES is resembling the typical one of a SR model at a critical point, assuming indeed a nearly continuous distribution. More importantly, the degeneracies of the Schmidt eigenvalues are not the ones generally expected for SR systems, signaling the appearance of the purely LR phases.

The explicit results are shown in Fig. 1. There it can be seen that for $\alpha \gtrsim 1$ the Schmidt eigenvalues ω_j (j labelling them starting from the highest one) composing the ES are arranged in well-separated multiplets. In particular when $|\mu| < 1$ (left panel, showing the case $\mu = -0.5$ and $\alpha = 3$), the dimension of the multiplets is even, as implied by the presence of two degenerate vacua (in the thermodynamic limit) $|\text{GS}\rangle$ and $|\text{GS}\rangle_0$ with different Z_2 fermionic parity, as recalled in the Appendix B (see also [56–58]).

Conversely, approaching the line $\alpha = 1$, the same multiplets tend to assume a continuous distribution and the decay of ω_j becomes much slower. Even more interestingly, also when $|\mu| < 1$ the even parity of the multiplets disappears, paralleling the presence of a unique (Z_2 -even) ground state $|\text{GS}\rangle$.

The absence of constraints on the parity of the Schmidt multiplets is also shown in Fig. 2, where the behaviour of the difference between the two highest Schmidt eigenvalues, called “Schmidt gap” [55], is reported for different values of μ and $L = 512$. We see that, approximately below $\alpha = 1$,

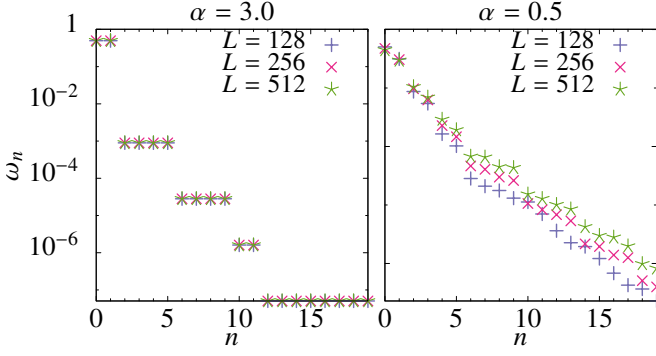


FIG. 1. Entanglement spectrum for the open LR paired chain in Eq. (1) with $\mu = -0.5$, different L , and for $\alpha = 3$ (left panel) and $\alpha = 0.5$ (right panel).

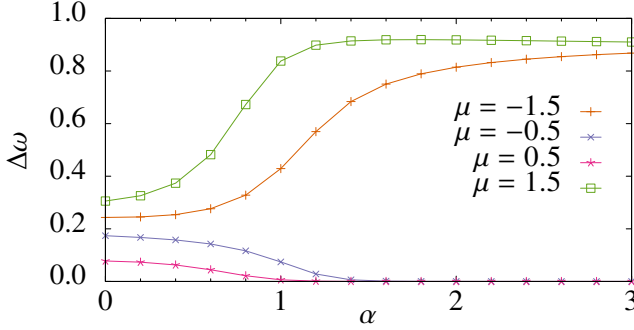


FIG. 2. Dependence on α of the difference $\Delta\omega$ between the two highest Schmidt eigenvalues (Schmidt gap) for the open LR paired chain in Eq. (1), different values of μ and $L = 512$. Notice that $\Delta\omega = 0$ if $\alpha \gtrsim 1$ and $|\mu| < 1$, as expected in the SR limit.

this quantity becomes nonvanishing for every μ .

In order to exclude that the nonzero values for the Schmidt gap below $\alpha = 1$ found in Fig. 2 are due to finite-size corrections, we show in Fig. 3 a finite-size scaling of the most unfavourable case in the former Figure ($\mu = 0.5$), done with the data for chains with lengths from $L = 60$ to $L = 512$. This scaling yields at $\alpha = 0.5$ a value $\Delta\omega \approx 0.05$ for the Schmidt gap, notably not far from the value at $L = 60$; this

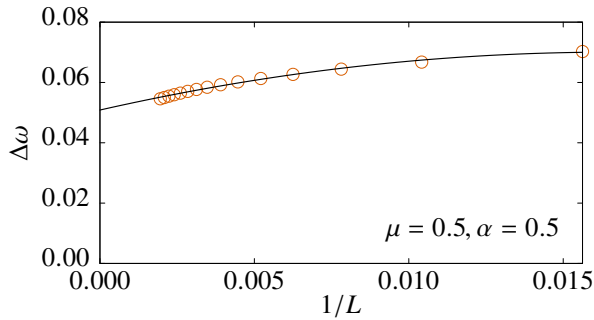


FIG. 3. Finite-size scaling of the Schmidt gap at $\mu = \alpha = 0.5$ (purple line in Fig. 2). The fitted value $\Delta\omega \approx 0.05$ strongly deviates from the much smaller values found at $\alpha > 1$ and the same μ (see the main text).

fact indicates the limited role of the finite-size effects for the $\Delta\omega$. Notice finally that $\Delta\omega \approx 0.05$ is much bigger than the value at $\alpha = 2$, where we obtain $\Delta\omega < 10^{-5}$.

The differences in the distributions of the ES in the range $|\mu| < 1$ and on the two sides of the line $\alpha = 1$ confirms the appearance of purely LR phases below this threshold. Even more interestingly, the presents results stress once more the deep difference between the latter phases and the SR ones. Indeed it is known [58, 59] (see also in the Appendix A) that, no matter the presence or the absence of interaction, two phases only (ordered and disordered) can be realized on a single SR Kitaev chain [52]. The disordered phase of this model, having a single ground-state $|\text{GS}\rangle$, displays no constraint on the number of Schmidt eigenvalues in each multiplet, so that multiplets with odd numbers of eigenvalues are also present. In particular, the minimum degeneracy for a multiplet is 1. On the contrary, the ordered phase is characterized by even Schmidt multiplets. In particular, the minimum degeneracy for a multiplet is 2.

The described SR picture is not realized instead in the LR phases at $\alpha < 1$. Indeed these phases display also single degeneracies in the Schmidt multiplets but, as inferred in the Sections III and VI, they are disconnected from the disordered phase of the SR Kitaev chain.

This peculiar behaviour for the ES parallels the violation of the area-law and the appearance of massive edge modes (when $\mu < 1$), as we will discuss in more detail in the Section VII. There the formal reasons for the failure of the discussion in [58] will be also analyzed.

V. FORMAL ORIGIN OF THE DEVIATIONS FROM TWC

In this Section we formalize the origin of deviations from the TWC that can occur in LR quantum systems, analyzing the hypothesis at the bottom of the TWC and their possible failure in the presence of LR Hamiltonian terms. The same analysis suggests that in general only some singularities in the energy spectrum can induce LR phases, while other ones preserve the SR phase content and more general the TWC.

We remember that, even if we are still dealing with superconducting phases, the main results of our discussion are not restricted to this set of systems. Moreover no limitations are implied on the dimensionality of the considered LR quadratic fermionic models.

A key to understand the TWC in any dimension is based on the classification of the topologically inequivalent continuous maps from the space of the lattice momenta $\mathbf{k} \in [0, 2\pi)$ (assumed to be a good quantum number, due to translational invariance in periodic systems) [60], to a suitable grassmanian manifold F , induced by the matrix Hamiltonian $H(\mathbf{k})$ ([1–5], [61] and references therein). These maps are defined univocally by some (sets of) integer numbers, generally called winding numbers. In general F has the form of a coset space G/\mathcal{H} , being G and \mathcal{H} some groups. In the absence of further symmetries, these manifolds are strongly constrained by

the discrete anti-unitary charge-conjugation and time-reversal symmetries.

For instance, in the particular case of spinless superconductors, as for the generalized Kitaev chains in Eqs. (1) and (3), it is useful to classify the windings of the unitary vectors $\hat{n}_{\mathbf{k}} = \frac{\mathbf{n}_{\mathbf{k}}}{|\mathbf{n}_{\mathbf{k}}|}$ such that $H(\mathbf{k})$ can be written as $H(\mathbf{k}) = |\mathbf{n}_{\mathbf{k}}| \hat{n}_{\mathbf{k}} \cdot \vec{\sigma}$, σ_i being the Pauli matrices. See [49] for their differential expressions in the one-dimensional D/BDI classes.

However, as discussed in the previous Sections, in the presence of LR Hamiltonian terms in real space, singularities for $H(\mathbf{k})$ and for its spectrum $\lambda(\mathbf{k})$ can occur. The behaviour of these divergences, say at a momentum \mathbf{k}_0 , strongly affects the definition of the winding numbers and the possible emergence of the LR phases. Indeed if $H(\mathbf{k}_0 + \epsilon)$ does not depend explicitly on ϵ (in one dimension if $H(k_0 + \epsilon) = H(k_0 - \epsilon)$), the singularities at \mathbf{k}_0 do not really spoil this definition (in the particular case above they are regularized dividing $H(\mathbf{k})$ by $|\mathbf{n}_{\mathbf{k}}|$: $\tilde{H}(\mathbf{k}) \equiv \frac{H(\mathbf{k})}{|\mathbf{n}_{\mathbf{k}}|}$), as well as of the related winding numbers. This is the case for $g_\alpha(k)$ in Eq. (4). In the following we will quote these divergencies as *first type divergences/singularities*. On the contrary, when $H(\mathbf{k}_0 + \epsilon)$ depends explicitly on ϵ (in one dimension if $H(k_0 + \epsilon) \neq H(k_0 - \epsilon)$), the path in the $H(\mathbf{k})$ manifold experiences not reabsorbable discontinuities. For this reason, the winding numbers of $H(\mathbf{k})$ that can define and classify the topology of the SR phases in every class of the TWC are now ill defined [62]. In this condition the TWC can generally fail and new (LR) phases can occur. In the following we will quote these divergencies as *second type divergences/singularities*. The different role to the two types of divergencies will be probed explicitly in Section VI.

Notably in [35], using an effective theory close to the critical lines (where conformal invariance is explicitly broken), it has been shown explicitly for the model in Eq. (1) that divergences in $H(\mathbf{k})$ of the second type induce directly the violation of the area-law for the Von Neumann entropy. Based on the discussion above, we are lead to think that this parallelisms holds generally in gapped LR fermionic systems.

Recently some attempts to define a topology for some LR fermionic systems appeared [37], exploiting the differential invariants in [49]. For the system in Eq. (1) these attempts led to mathematically inconsistent results for $\alpha < 1$: indeed semi-integer winding numbers have been obtained in this condition, in spite of the fact that, being measured on closed d -dimensional loops, winding numbers should assume only integer values [50, 51]. In this way the topology defined in terms of these winding numbers is mathematically not even well defined, as well as a connection between these numbers, calculated in the bulk, and possible edge excitations (see more details in Section VI and VII). However, in the light of our discussion, the mere appearance of winding numbers with fake semi-integer numbers (the same numbers instead well defined with integer values in the SR limit) can be interpreted as a clear physical diagnostic of new LR phases beyond TWC. This point will be discussed in better detail in Section VI.

The analysis above suggests that no new phase is expected in the presence of other singularities in higher-order deriva-

tives of the spectrum $\lambda(k)$, as for the Hamiltonians in Eqs. (1) and Eqs. (3) in $k = \pi$ at every finite $\alpha > 1$ (on the contrary, the same singularities has been found responsible of other LR effects, as explained in the previous Sections). In this way, a particular care appears required to evaluate the regime at $1 < \alpha < \frac{3}{2}$ for the Hamiltonian in Eq. (1), suspected in [37] to have LR nature. For more details, see Sections VI, VII A and VII.

We discussed above that in the SR limit topology can be encoded in some (sets of) windings number(s) induced by the mapping $\mathbf{k} \rightarrow H(\mathbf{k})$ itself [3, 4]. Exploiting a nonlinear σ -model description of these grassmanian manifolds, in the past literature the TWC has been obtain directly [1–4]. Indeed F is strongly constrained by the (anti-unitary) symmetries of the system under consideration, setting its topology class. Remarkably, the same approach implicitly addresses the stability of the phases of the SR topological insulators against the introduction of a onsite disorder; indeed the latter ingredient is explicitly assumed and encoded.

In the Appendix C we show instead the general failure, in the presence of (LR) singularities in $H(\mathbf{k})$, of the non linear σ -model construction leading to the TWC, at least as derived following the standard approach in [63–68]:

$$S_\sigma[Q] \sim C \text{Tr}(\nabla Q \nabla Q), \quad (7)$$

being Q a effective bosonic field and C a divergent constant in the LR phases.

We comment finally that, differently from SR models, in LR systems the effect of disorder could be expected more dramatic, spoiling the divergences in the energy spectrum that originate the LR phases and all the other described LR peculiarities. However this possibility will be ruled out in Section VIII B.

VI. TOWARDS A BULK CLASSIFICATION OF LR PHASES

In this Section we deal with the problem of classifying the purely LR phases of quadratic fermionic Hamiltonians. We perform the discussion mostly for one-dimensional systems. We discussed in Section V that winding numbers for the matrix Hamiltonians $H(k)$ are apparently not useful, being ill defined in the LR phases. The reason for that failure lies on the discontinuities encountered in the path on the $H(k)$ manifold as k varies in the Brillouin zone, e.g. in the correspondence of second type divergences (say at k_0), where $H(k_0 + \epsilon) \neq H(k_0 - \epsilon)$.

A different approach, primarily valid in the SR limit, is to consider the Berry phase

$$\Phi = i \int_{BZ} dk \langle u_k | \partial_k u_k \rangle, \quad (8)$$

gathered again as k varies along the Brillouin zone. The vector $|u_k\rangle$ is an eigenvector of $H(k)$ and the integral extends on the Brillouin zone. The same approach has been exploited in

[37] for the Hamiltonian in Eq. (1).

For sake of generalization and in order to identify ambiguity problems in the definition of a nontrivial LR topology, it is useful to discuss here the main step of the calculation for the models in Eqs. (1) and (3), both in the SR limit and in the LR one. In both of the regimes the same calculation proceeds in a very similar way.

We notice at first that, since $\langle u_k | u_k \rangle = 1$, we have that, if we fix $|u_k\rangle$ real, $\langle u_k | \partial_k u_k \rangle = 0$, unless $|u_k\rangle$ and/or $|\partial_k u_k\rangle$ are singular. In all the cases considered in the present paper $|u_k\rangle$ are well defined (finite), as well as the Bogoliubov transformations leading to them [27, 35], while the second possibility can be realized, being $|u_k\rangle$ discontinuous. This fact holds in the correspondence of second type singularities (say again at k_0):

$$|u_k\rangle = |v_{1k}\rangle + \theta(k - k_0)|v_{2k}\rangle, \quad (9)$$

or, equivalently,

$$|u_k\rangle = M_{k_0}(k) |v_k\rangle = (\mathbf{I} + \theta(k - k_0) N(k)) |v_k\rangle, \quad (10)$$

being $M_{k_0}(k)$ and $N(k)$ suitable matrices. Notice that $N(k)$ is continuous through k_0 : $N(k_0 + \epsilon) = N(k_0 - \epsilon) \equiv N(k_0)$, moreover we have continuity in k_0 for the energy $\lambda(k)$ of $|u_k\rangle$: $\lambda(k_0 + \epsilon) = \lambda(k_0 - \epsilon) = \lambda(k_0)$. The latter fact is central to assure the Berry phase to be well defined.

The matrix $M_{k_0}(k)$ transforms locally $H(k)$ where the singular point k_0 is encountered:

$$H(k_0 + \epsilon) = M_{k_0}(k_0 + \epsilon) H(k_0 - \epsilon) M_{k_0}^{-1}(k_0 + \epsilon) \quad (11)$$

so to assure that

$$\lambda(k) = \langle u_k | H(k) | u_k \rangle \quad (12)$$

varies continuously passing through k_0 . We stress that, in spite of the matrix $M(k)$, the nature of the Berry phase Φ is purely abelian in all the cases analyzed in this paper, since no degeneracy for the ground state occurs.

It is straightforward to show that, passing through k_0 from below, a Berry phase

$$\Phi_{k_0} = -\frac{\pi}{4} \langle v_{k_0} | M(k_0)^{-1} \partial_k M(k) |_{k_0} | v_{k_0} \rangle \quad (13)$$

is gathered. This expression can be easily evaluated writing $\theta(k - k_0) = \frac{1}{2} (1 + \text{sign}(k - k_0))$ and using the complex expression for the derivative of the sign(k) function:

$$\frac{\partial \text{sign}(k)}{\partial k} = i \pi \delta(k) \text{sign}(k). \quad (14)$$

Importantly, the calculation scheme described above works completely equivalent for the discontinuities occurring in SR systems and for the LR ones from the second type singularities. Exploiting the same scheme, it is easy to show that:

- for the Hamiltonian in Eq. (1) ($\alpha < \infty$, $\beta = \infty$), $|u_k\rangle = (\cos \theta_k, \sin \theta_k)$, with

$$\theta_k = -\frac{1}{2} \arctan \left(\frac{f_\alpha(k)}{(\mu + g_{\alpha=\infty}(k))} \right), \quad (15)$$

and $g_\infty(k) = \cos k$. Two discontinuities in $\sin \theta_k$ in $|u_k\rangle$ arise (at $k_{1,2}(\alpha)$) if $|\mu| < 1$ and for every α , because there the diagonal terms in Eq. (2) change sign. Passing through each of them from below, a $\frac{\pi}{2}$ phase is gathered. Indeed if $\alpha > 1$ they give rise to the total Berry phase $\Phi = \pi$ proper of the phase with massless edge modes. In these cases the matrix $M(k)$ in Eq. (10) around $k_{1,2}(\alpha)$ reads: $M(k) = (\mathbf{I} + \theta(k - \pi)(\sigma_x - \mathbf{I}))$. Another discontinuity in $\sin \theta_k$ appears if $\alpha < 1$, because of a second type singularity at $k = \pi$, responsible of the outcome of LR phases. More in detail, this is due to the behaviour of $f_\alpha(k)$: $f(k \rightarrow \pi^+) = \infty$ and $f(k \rightarrow \pi^-) = -\infty$. We find that, passing through $k = \pi$ from below, a $\frac{\pi}{2}$ phase is gathered if $\mu > 1$, while a $-\frac{\pi}{2}$ phase is gathered if $\mu < 1$. In these cases the matrix $M(k)$ in Eq. (10) reads, around $k = \pi$, $M(k) = \text{sign}(\mu - 1) \text{diag}(1, \text{sign}(\pi - k))$, the same found in [37].

Collecting all these partial results, it is found that $\Phi = 0$ if $|\mu| > 1$ and $\Phi = \pi$ if $|\mu| < 1$ when $\alpha > 1$, while at $\alpha < 1$ we obtain $\Phi = -\frac{\pi}{2} = \frac{3\pi}{2}$ if $\mu < 1$ and $\Phi = \frac{\pi}{2}$ if $\mu > 1$.

These findings show the presence of two purely LR phases at $\alpha < 1$ disconnected from the SR ones (having different values of Φ), as described in the Section II. Moreover, they indicate a nontrivial topology for (at least) one of them, see more detail in the following of the Section.

- for the Hamiltonian in Eq. (3) with $\alpha = \infty$ and $\beta < \infty$ (LR hopping, SR pairing), exploiting the expression for θ_k in Eq. (15), only phases with $\Phi = 0$ and $\Phi = \pi$ are found, also at $\alpha < 1$. Correspondingly, a qualitatively equal situation as for the Hamiltonian in Eq. (1) at $\alpha > 1$ takes place: massless edge modes are found if $\Phi = \pi$, while no edge mode at all if $\Phi = 0$. These results formalize our expectation that first type singularities, as from $g_\beta(k)$ ($f_\infty(k) = \sin k$ is regular), do not induce alone the LR phases. Indeed these singularities yield $M(k) = \mathbf{I}$.

- for the Hamiltonian in Eq. (3) with both finite α and β , we obtain:

– if $\alpha < \beta$, below $\alpha = 1$ we find, as μ varies, zones with $\Phi = \frac{\pi}{2}$ and $\Phi = \frac{3\pi}{2}$, as for the case $\alpha < \infty$ and $\beta = \infty$ (Hamiltonian in Eq. (1)). In particular, at fixed α the second zone occurs at smaller μ compared with the first one. Correspondingly, the same content of massive edge states is found;

– if $\alpha > \beta$, below $\alpha = 1$ we find, as μ varies, zones with $\Phi = \pi$ and $\Phi = 0$, as for the Hamiltonian in Eq. (3) with LR hopping only. Indeed here the contribution of the (first and second type) singularities at $k = \pi$ from $g_\beta(k)$ and $f_\alpha(k)$ effectively cancel each other. At fixed α , the first zone occurs again at smaller μ compared with the second one. If $\Phi = \pi$ massless edge modes are found, while no edge mode at all when

$\Phi = 0$.

– if $\alpha = \beta$, at $\alpha < 1$ we find, as μ varies, zones with α dependent Berry phases: $\Phi = -\pi K(\alpha)^2$ and $\Phi = \pi \left(1 - K(\alpha)^2\right)$, with $K(\alpha) = -\sin\left(\frac{1}{2} \arctan \frac{1}{\tan(\frac{\pi}{2}\alpha)}\right)$. Again at fixed α , the second zone occurs at smaller μ compared to the first one, in this regime massive edge states have been previously found, for the first time [28]. The contribution $\propto K(\alpha)$, due to the second type singularity, vanishes at $\alpha = 1$, as expected, while it tends to $-\frac{\pi}{\sqrt{2}}$ at $\alpha = 0$ (then the same values for Φ as in the first example above are recovered). Strikingly, the quantity Φ varies continuously in the range $0 \leq \alpha \leq 1$, so that apparently it cannot be assumed *a priori* as an order parameter to distinguish SR and LR phases (while it appears effective to discriminate between the LR phases). However, a way to remove this problem will be discussed just below. Moreover, the presence of these phases can be proven also by the ground-state degeneracy arguments in Section II B.

We notice that, as resulting from the discussed examples, differently from the SR systems, for LR ones the appearance of a nonzero Berry phases, does not imply the presence of edge states in general. For instance, in the first example (LR pairing only), in the regime $\mu > 1$ a phase $\Phi = \frac{\pi}{2}$ is derived, notably not reflecting in the presence of massive edge states, in spite of the fact that this value is different from $\Phi = 0$ proper of the vacuum beyond the edges. The difference stems from the second type divergence at $k = \pi$; this contribution is present for *all* the LR phases at every μ , indeed is exactly the one discriminating SR and LR phases.

The latter example indicates the necessity for a more specific criterium to link the Berry phase Φ with the possible presence of edge states and with their properties in LR phases. From all the analyzed examples, we can conjecture that, given a certain model having different LR phases with Berry phases $\{\Phi_i\}$, edge states occur in the correspondence of every $\Phi > \Phi_M \equiv \min \{\Phi_i\} \pmod{2\pi}$, or said in another way, whenever $\tilde{\Phi} = \Phi - \Phi_M > 0$. Indeed Φ_M appears as a common contribution only discriminating the LR phases from the SR ones. In this way, the quantity Φ would define instead an effective LR topology, directly related with the massive edge states. We leave as a central open problem to probe this conjecture.

The Berry phase approach, with the caveats discussed above, can be extended to classify LR phases of one-dimensional Hamiltonians with different symmetry content from the BDI class examined in this paper.

A similar ambiguity as for the Berry phase is found in the attempts to define a LR topology by *winding numbers*, as in Section V, again due to the primary difficulty to identify the *trivial* LR topology. Indeed, both for the Kitaev Hamiltonian with LR pairing only [37] and for the Hamiltonian with also LR hopping, the path of the vector \mathbf{n}_k (defined in Section

V) in the two LR phases, as k changes from 0 to 2π , is a semi-circle around the origin $(0, 0)$, ended by a jump between its open edges, the jump being a direct consequence of the LR divergences. The closed paths (considering also the jumps) in the two phases differ by a entire circle, a fact that allows to define again a nontrivial topology between them and to infer the appearance of massive edge states in one of them. However, it appears unclear how to identify exactly the LR phase with trivial topology and the other one with nontrivial topology and massive edge modes.

Summing up, the discussion of this Section confirms the potential effectiveness of the Berry phase and of the generalized winding numbers approaches to define a nontrivial topology in LR quantum systems, provided the proper primary identification of the *trivial* topology. In particular, this possibility concerns also LR quantum systems with higher dimensionality (for a review on the same methods applied to general SR topological insulators see for instance [61] and references therein), indeed no additional obstructions seem to appear in these conditions. The evaluation of the two approaches on specific higher dimensional examples (an issue also involving the problem to define LR topological numbers entirely in terms of *local* quantities/currents neglecting path discontinuities) deserves deep future attention in the opinion of the authors.

VII. FAILURE OF EDGE CHARACTERIZATION: WEAKENING OF BULK-BOUNDARY CORRESPONDENCE

In Section IV we explained that the distribution of the ES in the LR phases of the Hamiltonian in Eq. (1) below $\alpha = 1$ does not insert in the SR classification scheme derived in [58] for the one-dimensional BDI symmetry class. In this Section we formalize the origin of this deviation. This analysis will yield further information on other LR peculiarities, for instance the nature of the massive edge modes, their link with the bulk excitations and the asymptotic behaviour of correlation functions. Moreover the same analysis will appear suitable to be extended almost straightforwardly to other one-dimensional symmetry classes of the TWC.

A. On the behaviour of the LR correlation length

The discussions of the previous Sections induce a further investigation on the definition of correlation length in the LR systems.

In gapped SR systems the correlation length ξ is defined by the asymptotical exponential decay of the two-points correlation functions ($\phi(x)$ being a typical field of the considered model):

$$C(x) = \langle GS | \phi(x) \phi(0) | GS \rangle_{x \rightarrow \infty} \sim e^{-\frac{x}{\xi}}; \quad (16)$$

the same quantity is also known to diverge at the continuous critical points, where instead correlation functions decay algebraically. This fact, also paralleling the violation of the area-

law for the Von Neumann entropy [45, 46], is at the basis of the scaling hypothesis and of the effective RG description for critical phenomena (see e.g. [69]).

Forcing the SR definition in Eq. (16) to LR systems, the hybrid decay for correlation functions discussed in the Introduction does not yield, at every finite α , a finite correlation length for the model in Eq. (1): $\xi \rightarrow \infty$ for every μ and $\alpha < \infty$. However, this naive definition does not match with the fact that if $\alpha > 1$ the realized phases are continuously connected with the ones in the SR limit (see Section II); at the same time the area-law for the Von Neumann entropy is fulfilled (up to possible sub-logarithmic corrections, see e.g. [35]) and the ES spectrum is distributed as in the same limit. Moreover, at $\alpha > 2$ the critical theory at $\mu_c \pm 1$ is the $c = \frac{1}{2}$ conformal theory characterizing the universality class of the SR Ising chain, a fact implying that ξ stays finite outside criticality [70]. At variance, only when $\mu = \mu_c$ it holds $\xi = \infty$; correspondingly the exponentially decaying parts of correlation functions become algebraic too [35].

For these reasons and considering the direct analogy with the SR systems (where close to criticality it holds $S(l) \sim \ln \xi$ as $l \rightarrow \infty$, $S(l)$ defined as in Section II B [46]), it appears natural to consider the correlation length effectively divergent *only* when the area-law for the Von Neumann entropy is violated, at $\alpha < 1$, where new LR phases appear [71]. Instead in the region $\alpha > 1$, ξ should be considered effectively finite, in spite of the strict result from the SR definition in Eq. (16). In total, from Eq. (16),

$$\xi = \begin{cases} < \infty, & \text{if area-law is fulfilled;} \\ \infty, & \text{if area-law is violated.} \end{cases} \quad (17)$$

In the quasi-SR regime at $\alpha > 1$, ξ can be likely defined by the exponential decay of the correlations close to the critical separation R^* with the algebraic decay range. An heuristic justification for Eq. (17) is in the behaviour of the point R^* separating the exponential and the algebraic decay regimes of the correlation functions. Indeed analytical and numerical calculations for the Hamiltonian in Eq. (1) (see Section VIII) indicate that, as α increases at fixed μ , R^* also becomes larger, then the exponentially decaying regime, bounded by it, gets wider. On the contrary, the algebraic decay tail shrinks as α decreases (in particular, the absolute value of the correlation function in R^* also decreases rapidly), finally becoming practically irrelevant for the behaviour of the system at $\alpha \lesssim 1$.

The definition for ξ in Eq. (17) notably allows to interpolate continuously between the phases at $\alpha = 1$ and in the SR limit $\alpha \rightarrow \infty$.

B. Failure of the edge characterization for the ES

In this Section we analyze directly the failure, in the presence of LR Hamiltonian terms, of the ES characterization for the one-dimensional BDI symmetry class discussed in [58].

We stress (see Appendix A) that, although the approach in [58] considers the presence of interactions (while the Hamiltonians in Eqs. (1) and (3) are quadratic), this qualitative difference does not hold for a single Kitaev chain.

In [58] the discussion is based on the analysis of certain edge operators $Q_{R/L}$, able to induce the bulk transformations belonging to the invariance group of the considered Hamiltonian, at least involving the states with highest Schmidt eigenvalues (then more likely after a bipartition). In this way, the (anti)-commutation relations between the operators $Q_{R/L}$ and with the generators of the symmetries of the Hamiltonian are able to constrain entirely the ES, classifying without ambiguities the SR fermionic phases. The properties of the edge operators reflect the ones owned by the edges, in particular due to the possible presence of localized modes on them. For one-dimensional quantum systems, both fermionic and bosonic (as spin models, analyzed in [72]), this construction formalizes the so called bulk-boundary correspondence conjecture.

The demonstration of these results starts, in [58], showing first that local operations performed asymptotically far from the edges cannot change the (highest part of the) entanglement content of the considered system. A crucial property exploited at this first step is cluster decomposition for correlation functions:

$$\langle GS|O(x)O(0)|GS\rangle_{x \rightarrow \infty} \sim \langle GS|O(x)|GS\rangle \langle GS|O(0)|GS\rangle. \quad (18)$$

Therefore it is expected that fermionic systems violating the cluster decomposition property can display important deviations from the classification scheme suggested in [58]. A remarkable case of violation of cluster decomposition in spin models has been described recently in [73] (another possible example is mentioned in the Appendix B). However, cluster decomposition is respected for the Bogoliubov ground state of the LR Kitaev chains in Eqs. (1) and (3) [27, 28, 35].

The second step of the discussion in [58] is the explicit construction of the boundary operators, relying on a MPS-like approach (valid for both fermionic and spin models). This construction is valid again for the highest Schmidt eigenstates and it requires the finiteness of the correlation length ξ . In particular the error in the implementation of the Hamiltonian symmetry transformations on these states by operators involving l sites from the edges scales as $\sim e^{-\frac{l}{\xi}}$.

However, for our models, in the correspondence of the appearance of the LR phases, ξ effectively diverges, as explained in Section VII A. Moreover, related with the divergence of ξ and as required implicitly by the MPS-like approach used in [58], the fulfillment of the area-law for the Von Neumann entropy results a necessary condition for the validity of the edge operators construction: this ingredient is again not present at $\alpha < 1$ in the LR systems studied in the present paper. Notably a logarithmic violation of the area-law in a gapped extended regime proves already sufficient to determine important deviations from the SR picture. Similar or even more dramatic deviations are expected for instance in LR systems where the area-law is substituted by a almost volume-law, a set described for instance in [38]. This point deserves an important future effort in the opinion of the authors.

We stress finally that, as the present discussion should make clear, the failure of cluster decomposition seems to imply the area-law violation and the effective divergence of the correlation length; the opposite implication is instead not true in

general (as also exemplified by the SR critical systems): the area-law violation seems to indicate the failure of cluster decomposition in its *exponential* form only [74].

C. Nature of the massive edge states and weakened bulk-boundary correspondence

The analysis of the last Subsection helps to shed light on the nature of the massive edge states found for the LR Hamiltonians in Eqs. (1) and (3) [27, 28].

Indeed the failure of the discussion in [58], based on the action of the edge operators $Q_{R/L}$, suggests that, oppositely to the SR limit, the purely LR phases cannot be characterized entirely by their edge structure. Indeed symmetry operations on a bulk state cannot be represented faithfully by operations near the two edges. For the same reason, a certain bulk structure, for instance of the ES, does not reflect directly in the properties of the two edges (e.g. the presence of localized modes on them). In this sense we have a violation of the so-called bulk-boundary correspondence, at the basis of the TWC in the SR limit.

The picture defined above seems not to match entirely the discussion performed in the previous Sections on the LR paired Kitaev chain. Indeed there the appearance of massive edge states (with mass m) below $\alpha = 1$ and at $\mu < 1$ has been found to parallel a nonvanishing Berry phase calculated in the bulk, in turn predicting a nontrivial LR topology. However, in this situation, in spite of the double edge localization for the wavefunction of the first excited state (with positive energy) $|m\rangle$, no real distinction between left and right edge modes can be made, oppositely to the SR limit. Roughly speaking, below $\alpha = 1$ the two edges are so correlated each others and with the bulk, that a rigorous definition of localized modes on each of them, distinguished from the bulk dynamical excitations, is not allowed any longer, not even in the thermodynamic limit. Such an important correlation is testified by the algebraic decay tails of the edge wavefunctions, strongly dominating at $\alpha \lesssim 1$ [27, 28]. In turn, the relevant overlap of the latter tails induced [28] to conjecture an hybridization mechanism of the SR Majorana modes, responsible for the appearance of the massive edge ones $|m\rangle$, in analogy with the situation occurring at finite sizes in the SR limit [42].

The described situation, apparently peculiar of the LR quantum systems, can be quoted as *weakened bulk-boundary correspondence*.

Exploiting the Bogoliubov construction of the bulk states and reviewing the standard construction of the edge states above $\alpha = 1$ [75], in the Appendix D we show the correctness of the picture described above, inferring in particular that the nonzero mass for $|m\rangle$ at $\alpha < 1$ is in a one-to-one correspondence with the impossibility to define from $|m\rangle$ two states localized separately on the left-hand and the right-hand edges. The same result implies in itself the failure, for our LR models, of the edge characterization in [58] and of the bulk-boundary correspondence valid for SR systems, in favour of the described weakened version.

We close the present Subsection noticing that the important

correlations between edge and bulk dynamics, resulting in the nonvanishing mass m for the analyzed one-dimensional examples, could indicate the absence of edge conductivity for LR topological insulators with dimensionality bigger than 1, where correlations are even enhanced. In our opinion, it is highly worthy to probe this conjecture by future investigations.

VIII. STABILITY OF THE LR PHASES

In this Section we investigate the stability of the LR phases at $\alpha < 1$ of the Hamiltonian in Eq. (1), against finite-size effects and local disorder. The present specific study appears easily generalizable to other LR non interacting models, also in higher dimensions.

A. Finite size stability

In the previous Sections we commented that the deviations from TWC occurring in LR quantum phases of the models in Eqs. (1) and (3) are due to the action of the singular states at $k = \pm\pi$, where the divergences in the spectrum $\lambda(k)$ appear for $\alpha < 1$. A natural general question at this points is how and to what extent the possible LR phases escaping the TWC in the thermodynamic limit can also occur in finite-size LR systems, where the divergences encoded by the singular states are smeared. This stability is clear in the analysis of the previous Sections, where numerical data for finite chains are reported, however a more formal justification would be desirable. The same question is also relevant for current experiments on LR systems, realized by trapped ions arrays, where very limited sizes (30-40 sites) can be reached only (see e.g [40, 41]).

The finite-size stability of the LR phases of the Hamiltonian in Eq. (1) can be understood for instance analyzing the behaviour for different values of L of the quantity [35, 75]:

$$m_\alpha(\mu) = \lim_{R \rightarrow \infty} \sqrt{\det G_{R,0}(\alpha, \mu)}, \quad (19)$$

where

$$\det G_{R,0}(\alpha, \mu) = \det \left[\delta_{R,0} + 2 \langle GS | a_R^\dagger a_0^\dagger + a_R^\dagger a_0 | GS \rangle \right]. \quad (20)$$

This parameter characterizes when $\alpha \rightarrow \infty$ the paramagnetic-(anti-)ferromagnetic quantum phase transition of the SR Ising model. Indeed in the same limit $m_\alpha(\mu)$ coincides [75] with the modulus of the average longitudinal magnetization

$$|\langle \sigma_x \rangle| = \lim_{l \rightarrow \infty} \sqrt{|\langle \sigma_i^{(x)} \sigma_{i+l}^{(x)} \rangle|} \quad (21)$$

of the SR Ising model: in particular it has non vanishing values when $|\mu| < 1$ only. The same parameter assumes a nonzero value below $\alpha = 1$ and at $\mu < 1$, in the LR phase with massive edge modes (see more details in the Appendix B).

In Fig. 4 we plot $m_\alpha(\mu)$ at $\mu = -3.2$ and various L and α .

We find that $m_\alpha(\mu) \neq 0$ if $\alpha \lesssim 1$ (the extension of the zone where $m_\alpha(\mu)$ change value is depending on L , as common also in SR systems), suggesting that the phase with massive edge modes survives also in the presence of important finite-size effects, smearing the singular states.

The same result is obtained analyzing the mass of the edge modes in the regime $|\mu| < 1$ and for varying α around the line $\alpha = 1$ [27]: this mass displays a qualitatively equal behaviour as $m_\alpha(\mu)$.

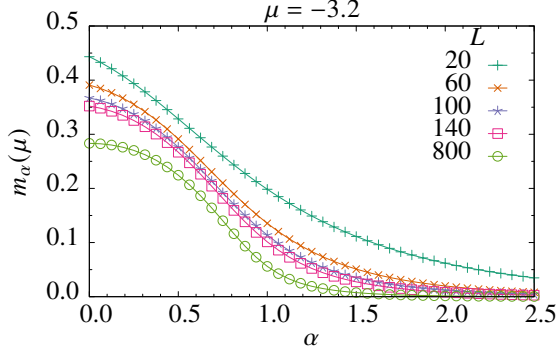


FIG. 4. Plot of the quantity $m_\alpha(\mu)$ in Eq. (19) for $\mu = -3.2$ and for different values of L and α .

The behaviour of $m_\alpha(\mu)$ can be understood in a better detail analyzing how the divergences at $k = \pm\pi$ develop in the matrix Hamiltonian in Eq. (2), in particular from the contribution by $f_\alpha(k)$. This can be done following the evolution with L and for different $\alpha < 1$ of the parameter

$$\mathcal{A} = \frac{|f_\alpha(\pi + \frac{\pi}{L}) - f_\alpha(\pi + \frac{3\pi}{L})|}{|\cos(\pi + \frac{\pi}{L}) - \cos(\pi + \frac{3\pi}{L})|}, \quad (22)$$

measuring the ratio between the differences of the functions $f_\alpha(k)$ and $\cos(k)$ calculated in the closest point to $k = \pi$, that means $k = \pi + \frac{\pi}{L}$, and in the second closest one, $k = \pi + \frac{3\pi}{L}$. We see in Fig. 5 that, at fixed L , \mathcal{A} rapidly increases as α decreases from 1 and, even in the most unfavourable case $\alpha \rightarrow 1$, we obtain $\mathcal{A} = 10$ if $L \approx 40$. The same threshold value for \mathcal{A} is obtained approximately at $L = 20$ if $\alpha = 0.5$. This behaviour means that for every $\alpha < 1$ the singularity in $\lambda(\pi)$ develops very rapidly with L increasing, making effective, already at limited sizes, the singular dynamics at the basis of the purely LR phases.

The stability of the phases below $\alpha = 1$ can be also clarified on the basis of general considerations about static correlation functions. All these quantities can be constructed from the two point ones $g_1(R) \equiv \langle a_R^\dagger a_0 \rangle$ and $g_1^{(\text{anom})}(R) \equiv \langle a_R^\dagger a_0^\dagger \rangle$ by Wick's theorem. The narrow transient regime around the distance R^* where the change occurs between the exponential and algebraic decay increases with α (as well as the magnitude of the correlation functions at R^*), so that, when $\alpha \rightarrow \infty$, the purely exponential decay, holding for SR gapped systems, is recovered. Oppositely, when $\alpha \lesssim 1$; the exponential decay regime is nearly absent; correspondingly the n -point correlations become more important.

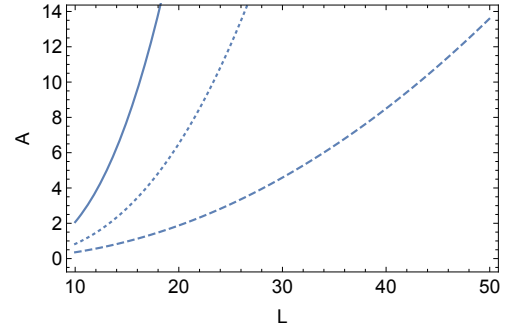


FIG. 5. Plot of the ratio \mathcal{A} in Eq. (22) for L varying and $\alpha = 0.99$ (dashed line), $\alpha = 0.5$ (dotted line), and $\alpha = 0.2$ (continuous line).

We plot in Fig. 6 (left panel) the behaviour of $g_1(R)$, for $\mu = -5$, $\alpha = 1.5$ and $\alpha = 3$, and for various system sizes L . We see that, decreasing L from $L = 300$, the algebraic tails become even and even shorter, while the exponential part remains practically stable. In particular the point $R^* \approx 20$, where the change between the two decay behaviours occurs, stays fixed. A qualitatively same behaviour is found for $g_1^{(\text{anom})}(R)$. When L reaches the length $L \approx R^*$, the algebraic tail disappears and the exponential part remains only, as for SR systems.

Conversely, decreasing α at fixed L , also R^* decreases [27, 28, 35]. In particular, as visible in Fig. 6, R^* becomes very small in comparison with L , so that the decay is purely algebraic. The described behaviour holds qualitatively no matter the values of μ and α and it has been probed also for other LR models (e.g. the LR Ising model [28]).

From the discussion above, it turns out that the size R^* gives the natural scale for the appearance of the LR physics. This means that when $L \lesssim R^*$ the system, even if described by an Hamiltonian with LR terms, is practically indistinguishable from its SR counterpart and to speak about LR physics has no mean in this condition. There $\mathcal{A} \lesssim 1$.

The present analysis justifies a posteriori the behaviour observed in Fig. 4 for $m_\alpha(\mu)$, suggesting the stability of the regime with massive edge modes at $\alpha < 1$, up to very small sizes $L > R^* \rightarrow 0$.

More in general, we can infer that possible LR phases escaping the TWC remain stable at finite-sizes, in spite of the fact that the origin of the deviations from TWC, the singular dynamics, is mathematically well defined in the thermodynamic limit only. In this way, the same phases are expected to be probable in current experiments, where only limited sizes are reachable.

B. Stability against local disorder

We inferred in Section V the failure of the σ -model construction, valid for SR systems, in LR free models at small enough α , due to a type of divergence in their energy spectrum at some momenta. The same construction encodes the effect of a local disorder and it allows to derive directly the

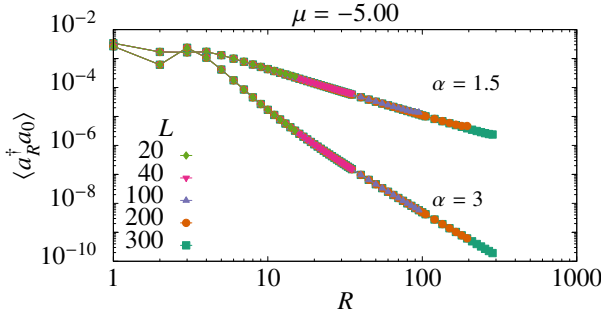


FIG. 6. Static correlation functions $g_1(R) \equiv \langle a_R^\dagger a_0 \rangle$ in log-log scale for the LR paired Kitaev chain in Eq. (1), for $\mu = -5$, $\alpha = 3$ (lower lines) and $\alpha = 1.5$ (higher lines), and different lengths L .

TWC (see e.g. [4]).

This result is somewhat counterintuitive, since disorder could be expected instead to smear and/or localize the divergences in the spectrum (and also the ones in its higher order derivatives), spoiling all the LR features.

In this Subsection we infer that this possibility can be ruled out, at least for free one-dimensional LR models as in Eqs. (1) and (3).

We consider the application of a local disorder. For our discussion we rely on a previous study of disorder on a spinless fermionic chain with LR hopping, performed in [76]. Further and more specific studies on the localization of edge modes in the presence of disorder have been performed in [37]. We also assume a disorder uniformly and randomly distributed in an interval $[-\frac{\eta}{2}, \frac{\eta}{2}]$, with standard deviation $\sigma \propto \eta$.

In momentum space the effect of the disorder Hamiltonian term H_D is to mix the quasiparticles at $H_D = 0$, possibly resulting in a localization of them in a limited region of the entire system. This mechanism is efficient for the singular states if the magnitude of the disorder $m_D \sim \frac{\sigma}{L^{1/2}} \sim \frac{\eta}{L^{1/2}}$ is at least comparable with the distance between the energy levels at $k \approx \pi$, $\delta\lambda \sim \frac{\eta}{L^{(\alpha-1)}}$, as shown rigorously in [76]. In the light of the two scaling laws for m_D and $\delta\lambda$, it is straightforward to infer that at $\alpha < \frac{3}{2}$ localization does not hold for the singular states, then the typical disorder in Eq. (C2) does not spoil the LR phases below $\alpha = 1$.

On the contrary the states far from $k = \pi$ are instead generally localized by H_D [76]: for this reason the disorder can even increase the relative importance of the role played by the singular states.

IX. SIMILAR ENTANGLEMENT BEHAVIOUR IN THE LONG-RANGE ISING MODEL

We inferred in the previous Sections that in gapped non-interacting fermionic systems the appearance of the area-law violation for the Von Neumann entropy and the related peculiar behaviour of the ES, as well as the resulting effective divergence of the correlation length, signals new purely LR phases, induced by a singular dynamics. We would like to probe now this picture on other LR systems, for instance spin models or

interacting fermionic ones. For this reason, we focus in this Section onto another paradigmatic LR system, the LR Ising model, recently studied theoretically [28, 30, 36] and experimentally in [40, 41]. In particular we discuss the behaviour of the ES.

A. Phase diagram and ground-state properties

The Hamiltonian of the LR Ising antiferromagnetic chain reads:

$$H_{\text{LRI}} = \sin \theta \sum_{i=1}^{L-1} \sum_{\ell=1}^{L-i} \frac{1}{\ell^\alpha} \sigma_i^{(x)} \sigma_{i+\ell}^{(x)} + \cos \theta \sum_{i=1}^L \sigma_i^{(z)}. \quad (23)$$

Recently this Hamiltonian has been simulated experimentally by atoms in cavity, with a coupling to photonic modes or by trapped ions coupled to some degrees of freedom related with their motion in real space. The so-obtained interactions decay algebraically with an exponent tunable in the range $\alpha \lesssim 3$ [40, 41, 77].

For the same model in Eq. (23), studying the Von Neumann entropy and the energy spectrum, obtained by DMRG calculations, in the range of parameters $0 < \theta < \frac{\pi}{2}$ (for $\frac{\pi}{2} < \theta < \pi$ the phase diagram is mirrored) and $0 < \alpha < \infty$, it has been shown in [30] that a quantum phase transition separating the antiferromagnetic and the paramagnetic phases survives for all the finite $\alpha \gtrsim 1$.

At variance, below this approximate threshold, a new phase arises on the paramagnetic side, bounded from above by a transition with non vanishing mass gap and having preserved spin-flip (along the \hat{x} axis) Z_2 symmetry (a unique ground-state appears in the DMRG spectrum). There a logarithmic violation of the area-law for the Von Neumann entropy has been found [28, 30]. In spite of the limited sizes achieved by DMRG ($L < 150$), in [30] the same violation has been probed also by finite size scaling, confirming it as not originated from finite size effects.

At every $\alpha < \infty$ in [28, 30] an hybrid decay for the static correlations is found, as for the LR Kitaev chains.

Finally, the model reduces at $\alpha \rightarrow 0$ to the so-called Lipkin-Meshkov-Glick (LMG) model, obtained by a zeroth-order mean field projection. This identification has been used so far to investigate the entanglement content and its dynamics after quenches [31, 78, 79]. The study of the same model allows to conclude the extensivity of the ground state energy in the LR phase on the paramagnetic phase [80], so that no Kac rescaling [11] is required to define rigorously the thermodynamic limit. The same result can be inferred by direct calculation in the limit $\theta \rightarrow 0, \pi$ [81].

The described phase diagram is depicted in Fig. 7, where the critical semi-line is reported, as well as the violation of the area-law. The antiferromagnetic and the paramagnetic phases at $\alpha > 1$ are denoted by the symbols AM and PM1 respectively, while the LR phase on the paramagnetic side at $\alpha < 1$ is quoted as PM2.

Similarly as for the LR Kitaev chains, a singular dynamics has been recently shown [36] responsible for the breakdown

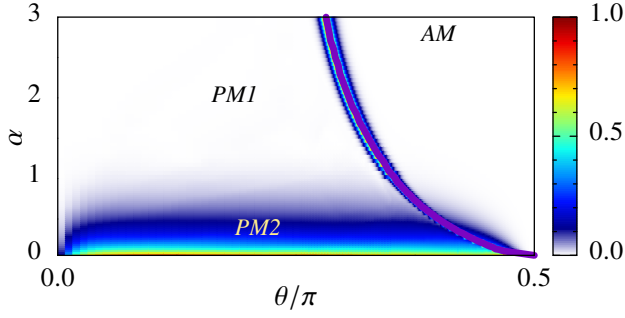


FIG. 7. Phase diagram of the LR Ising model in Eq. (23), derived analyzing the area-law deviation for the Von Neumann entropy. We report in particular the quantity c_{eff} defined in Eq. (6). The purple semi-line is critical, there the mass gap vanishes. The narrow zone close to the line $\theta = 0$ is left white since not investigated, due to a DMRG instability. The antiferromagnetic and the paramagnetic phases at $\alpha > 1$ are denoted by the symbols AM and PM1 respectively, while the LR phase on the paramagnetic side at $\alpha < 1$ is quoted as PM2.

of the conformal invariance along the antiferromagnetic-paramagnetic quantum phase transition at small enough α (from a critical α debated in the range $1 \leq \alpha \leq 2$ [28, 30]).

The LR spin Hamiltonian in Eq. (23) can be mapped via a Jordan-Wigner transformation to an interacting LR interacting fermionic chain, shown in the Appendix E. While the energy spectra and the phase contents of the two open models are in a one-to-one correspondence, since the Jordan-Wigner transformation is nonlocal, the knowledge of the properties (entanglement content, edge properties) of the Hamiltonian in Eq. (23) does not help to shed light on the same properties of Eq. (E2) (see e. g. [82]). For instance, as shown in [28], the phase of the fermionic Hamiltonian at $\alpha < 1$ and in anti-ferromagnetic regime is characterized by edge localization of the lowest eigenstates with nonzero energy, spread instead in the bulk at higher values of α . Therefore again massive edge states appear, similar to the ones for the LR paired Kitaev chain. This facts parallels and confirms the existence of a new phase for the LR Ising model.

B. Entanglement spectrum and failure of MPS-based bulk-boundary classifications

We study in this Subsection the ES for the LRI chain after an half-chain cut and we compare it with the results in the SR limit. We also assume open boundary conditions.

Exploiting a MPS based boundary operators approach, similar to the one discussed in Section VII for the BDI one-dimensional Hamiltonians, in [72] it has been shown that for a purely Z_2 symmetric spin chain only two disconnected phases with preserved Z_2 spin-flip symmetry can be found, having the ES contents qualitatively equal to the two phases of the SR Ising Hamiltonian [53] [83]) or the open SR Kitaev chain (see Section VII). More in detail, one (ordered) phase displays a distribution for the ES constrained in Schmidt multiplets with even degeneracy, while the other (disordered) one has no con-

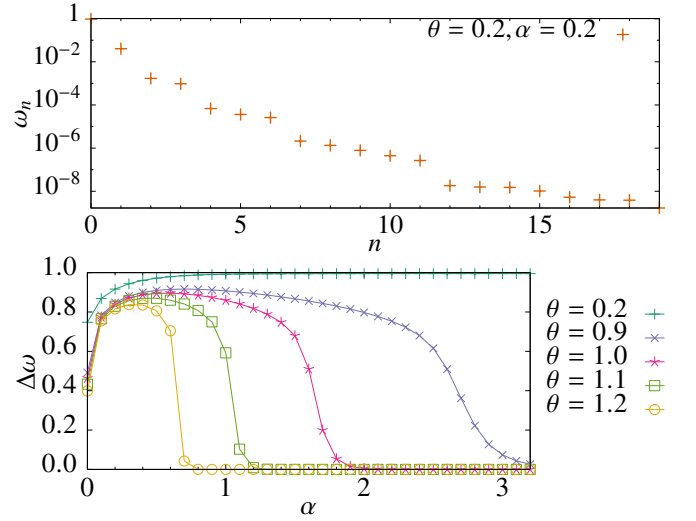


FIG. 8. Upper panel: distribution of the ES (Schmidt eigenvalues) for the LR Ising chain with $L = 100$, in the point belonging to the LR phase at $\theta = \alpha = 0.2$. Lower panel: behaviour of the Schmidt gap $\Delta\omega$ as a function of α and for different θ . Notice that $\Delta\omega$ is always nonvanishing along the line $\theta = 0.2$, since the same line never enters in the paramagnetic side where the LR phase occurs. Moreover, approaching $\theta = 0$, $\Delta\omega$ increases, as expected.

straint on the same distribution, in particular a minimum multiplet degeneracy equal to 1.

Notably the latter situation is found also in the LR phase below $\alpha = 1$ on the paramagnetic side, as visible in Fig. 8. In particular, in the lower panel, we display the behaviour of the Schmidt gap (as defined in Section IV) $\Delta\omega$ for $L = 100$ and different values for α , finding a non closure for it if $\alpha < 1$. We also see that $\Delta\omega$ at $\alpha \rightarrow 0$ increases as θ gets closer to 0 and the LR phase gets far from its bound with the anti-ferromagnetic phase.

For this reason, similarly as for the LR Kitaev chains, we find that the ES distribution found in the LR phase at $\alpha \lesssim 1$ escapes the SR classification in [72]: indeed again we find, a second (LR) phase with realized Z_2 spin flip symmetry and no constraint on the Schmidt multiplets. This behaviour remains up to the LMG limit at $\alpha \rightarrow 0$. Since the main steps in the discussion of [72] follow the same logic as in [58], from Section VII we can conclude that the reason of this deviation is again the violation (even if only logarithmic) of the area-law for the Von Neumann entropy and the related effective divergence of the correlation length, spoiling MPS-based edge operators approaches.

X. CONCLUSIONS

In this paper we formalized the emergence of new types of bulk insulators in the presence of long-range Hamiltonian terms, not included in the classification of the short-range topological insulators, the so-called "ten-fold way classification". The reasons of these deviations are analyzed, first studying specific one-dimensional examples, later on focus-

ing on the general structure, in any dimension, of long-range non interacting fermionic Hamiltonians.

The new phases are found originated from a particular type of the divergences occurring in the thermodynamic limit due to the long-range couplings: if the long-range nature is important enough, the same divergencies spoil some continuity hypothesis (mainly on the bulk energy spectrum) at the basis of the ten-fold way classification, determining its breakdown. Related to this fact, a topology can be still defined, at least for one dimensional systems, by winding numbers or Berry phase approaches, provided a proper identification of the long-range contributions to them.

From a many body point of view, the central ingredient for the appearance of purely long-range insulating phases seems to be the violation of the area-law for the Von Neumann entropy. Notably a logarithmic (soft) violation is suggested already sufficient for the considered one-dimensional examples. In the same models, the emergence of the long-range regimes deeply reflects in the behaviour of another entanglement indicator, the entanglement spectrum, whose analysis also allowed to find further evidences of deviations from the short-range scenario, as well to reconsider critically the link between bulk and edge dynamics. Moreover, also in the light of the hybrid (exponential plus algebraic) decay behaviour found previously for the static correlation functions of the considered models, the area-law violation induced to rediscuss the concept of correlation length in long-range systems. In particular, in spite of the mentioned algebraic decay tails for correlations, this length appeared to be defined as infinite only in the presence of area-law violation.

The stability of the long-range phases against finite-size effects and local disorder is also discussed, showing notably that current trapped ion techniques already allow to reach sufficient system sizes to guarantee their simulatability. Another notable result of the latter discussion is that disorder can even strengthen the effects of the long-range Hamiltonian terms, instead of smearing them, as it could be naively expected.

Concerning the edge properties of the purely long-range phases, the analysis of the entanglement spectrum strongly suggested the partial failure, at least in one space-dimension, of the bulk-edge correspondence, valid instead for short-range topological insulators, due to the strong correlations between bulk and edges dynamics. Said in another way, the presence itself of modes localized separately at each edge and distinguished from the bulk excitations appeared difficult to sustain. However, the parallelism between the appearance of massive edge states and of nontrivial Berry phases and winding numbers (although defined with particular caveats, such to identify properly the long-range contributions) suggested the emergence of a weakened form of bulk-boundary correspondence, peculiar of long-range quantum systems.

Finally, the possible extension of some results and ideas for the free long-range insulators has been probed on a paradigmatic example of spin model, the long-range Ising chain. Again important deviations from the structure expected for the short-range spin chains are identified in the entanglement con-

tent; consequently the limitations of bulk-boundary (tensor-network based) approaches to classify long-range spin models is inferred.

Natural developments of the present work are: *i)* the classification of the long-range topological phases for quantum systems with arbitrary dimensionality. The analysis of the conditions for the emergence of massive edge modes seems to be a promising approach; *ii)* the corresponding investigation of the nature of the edge states in long-range fermionic systems with dimensionality bigger than one, in order to probe the bulk-boundary correspondence (also following the logic in [84, 85]). Interestingly from the experimental and technological points of view, this issue also concerns the possible absence of edge conductivity, present instead for short-range topological insulators; *iii)* the generalization to interacting long-range models, also exploiting entanglement indicators (for instance, in the short-range limit the entanglement spectrum proved effective also in the presence of explicit interactions [58]); *iv)* the study of the effects of stronger deviations from the area-law for the Von Neumann entropy, for instance assuming a (almost) volume-law scaling, as in the set of systems investigated in [38]; *v)* the understanding of the role of disorder on the singular dynamics in interacting long-range systems, as for the long-range Ising model. There effects of many-body localization [86] are expected to play a relevant role; *vi)* the identification of a general scheme for the experimental detection of the long-range phases, for instance based on direct measurements of the entanglement or of some topological invariants, e.g. by imaging techniques; *vii)* the stability of the long-range (topological) phases against a nonzero temperature, classically correlating the parts of the considered quantum systems.

Note – Close to the conclusion of the present manuscript, another paper appeared [87], investigating the structure of the edge modes in long-range Kitaev chains. The hybridization mechanism, conjectured in [28] and leading to the massive edge modes at $\alpha < 1$, is rigorously shown in the limit $\alpha = 0$. The same mechanism, possible thanks to the algebraic decay tails (overlaps) of the Majorana edge wavefunctions [27, 28], agrees with the weakening of the bulk-boundary correspondence outlined in our paper.

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Appendix A: On the classification of topological insulators in short-range systems

In this Section we briefly review some aspects of the gapped fermionic systems (bulk insulators) with short-range (SR) Hamiltonian terms. Most of the known results about this topic concern fermionic systems without any interaction: their classification in any dimension is known as "ten-fold way" (TWC) [1–5]. This name derives from the fact that such systems are classified, according to their invariance (or not) under time reversal and particle-hole anti-unitary symmetries (the anti-unitarity deriving from the requirement of stability against local disorder), in ten different classes. Indeed these symmetries have been realized to impose important constraints on the momentum space matrix Hamiltonians $H(\mathbf{k})$ of the free insulators, in particular on their possible topological properties. The latter features are encoded for instance in some nonvanishing numbers that characterize the paths of $H(\mathbf{k})$ (or of their eigenstates) in matrix (vector) spaces as \mathbf{k} varies on the corresponding Brillouin zones. These numbers, defining a nontrivial topology, quantize the edge conductivity encountered in some insulators, indeed called "topological".

Without entering in the details of the classification, we recall below some properties at the basis of the TWC and some important consequences that will be rediscussed in the main text:

i) according to TWC, states with nontrivial topology host at the edges some *massless* modes, interpolating between the topological bulk and the trivial vacuum beyond the edges. Actually a well-stated approach to obtain the TWC [3, 4] is based on the characterization of the massless edge modes protected against Anderson localization in the presence of a random disorder not spoiling the symmetries of the bulk. The rationale at the bottom of this approach is a so-called (holographic) bulk-boundary correspondence, linking bulk and edge properties. The formalization of this correspondence for a specific one-dimensional class of the TWC (BDI) has been discussed in the main text. The presence of massless edge modes distinct from the bulk excitations is at the bottom of the edge conductivity observed in topologically nontrivial (bulk) insulators;

ii) states with different topology are separated by second-order quantum phase transitions, where the mass gap closes;

iii) the TWC is characterized by SR (exponential) correlation functions and SR entanglement [10], in particular by the fulfillment of the area-law for the Von Neumann entropy between two disconnected parts of the considered system.

One-dimensional BDI class: Kitaev chain

A notable symmetry class of the TWC is the BDI class, where the Hamiltonians are invariant both under time reversal symmetry and particle-hole symmetries, realized respectively by the anti-unitary transformations $\mathcal{U}_T = K U_T$ and $\mathcal{U}_c = K U_C$, being K the complex-conjugation operator. In particular in this class it holds $U_T^2 = U_C^2 = 1$.

Restricting to one-dimensional fermionic systems, the paradigm of the BDI class is the SR Kitaev chain [42]:

$$H_{\text{kit}} = - \sum_{j=1}^L \left(a_j^\dagger a_{j+1} + a_j^\dagger a_{j+1}^\dagger + \text{h.c.} \right) - \mu \sum_{j=1}^L \left(n_j - \frac{1}{2} \right). \quad (\text{A1})$$

In Eq. (A1), a_j is the operator destroying a (spinless) fermion in the site $j = 1, \dots, L$, being L the number of sites of the chain.

iv) this open Hamiltonian hosts two phases [52], characterized respectively by the winding numbers $w = 0$ and $w = 1$ of the first homotopy class [50, 51] of the map $k \rightarrow \mathbf{n}(k)$, where $\mathbf{n}(k)$ is such that the matrix Hamiltonian in momentum space is written as $H(\mathbf{k}) = |\mathbf{n}(\mathbf{k})| \hat{\mathbf{n}}(\mathbf{k}) \cdot \vec{\sigma}$ and $\hat{\mathbf{n}}(\mathbf{k}) = \frac{\mathbf{n}(\mathbf{k})}{|\mathbf{n}(\mathbf{k})|}$ (see the main text). In the (disordered) phase with $w = 0$ a unique ground-state $|\text{GS}\rangle$, eigenstate of the Z_2 fermionic parity (with even parity), occurs (see e.g. [47]). This parity is defined in general on the number of fermions $\langle \hat{F} \rangle = \langle \sum_{i=1}^L a_i^\dagger a_i \rangle$ in a certain state.

The second (ordered) phase with $w = 1$ is characterized by the presence of two Majorana (massless) edge modes at its ends, exactly due to its nontrivial topology. Thanks to the presence of the massless edge modes, two ground-states, $|\text{GS}\rangle$ (defined just above) and $|\text{GS}\rangle_o$, are present in the thermodynamic limit, having different Z_2 fermionic parity. In particular $|\text{GS}\rangle_o = \eta_0^\dagger |\text{GS}\rangle$ has odd parity; the fermionic operator $\eta_0 = \eta_L + i \eta_R$ is constructed by the ones related with the two massless (Majorana) edges modes $\eta_{\{R,L\}}$. The states $|\text{GS}\rangle$ and $|\text{GS}\rangle_o$ are degenerate in energy in the thermodynamic limit, exactly because the edge modes are massless. However, no spontaneous symmetry breaking, indicated by a local order parameter, occurs (see e.g. [47, 58]).

The two phases also correspond, via Jordan-Wigner transformation, with the ones of SR Ising model, discriminated by (the modulus of) the expectation value of the average longitudinal magnetization $|\langle \sigma_x \rangle| = \lim_{l \rightarrow \infty} \sqrt{|\langle \sigma_i^{(x)} \sigma_{i+l}^{(x)} \rangle|}$, a local parameter [75, 90]. In turn this parameter signals the behavior of the two phases under the Z_2 (σ_x) spin-flip symmetry, in the two cases realized and spontaneously broken respectively. The spin-flip Z_2 symmetry and the Z_2 fermionic parity of the open SR Kitaev chain are related by the following relation [47]:

$$(-1)^{\hat{F}} = \prod_{i=1}^L \sigma_i^{(x)}. \quad (\text{A2})$$

v) Beyond the model in Eq. (A1), infinite one-dimensional BDI phases can be constructed, with higher winding numbers belonging to the set Z . For instance a phase with $w = n$ can

be obtained connecting n SR Kitaev chains in the Majorana (ordered) phase to form a ladder. The topological quantum number $w = n$ corresponds then with the number of pairs of dangling Majorana fermions at the edges of the ladder [58, 59]. This identification can be proved more in general for every one-dimensional free fermionic system by a theorem on adjacency matrices (see e. g. [91]). The same theorem is a direct formalization of the bulk-boundary correspondence.

As said at the beginning of this Appendix, TWC holds in the absence of interactions between the fermions of the considered insulator. However for some class of the TWC the stability of the edge modes has been proven also against the introduction of some explicit interactions [58, 92]. In these conditions, the definition and identification of nontrivial topologies are a more difficult task and results are available in some cases only.

Interacting one-dimensional BDI classes have been classified in [59] (using K -theory) and in [58] analyzing the entanglement spectrum (ES). The latter quantity is defined as the set of (Schmidt)-eigenvalues of the reduced density matrix ρ_B of a part B of the considered quantum system after a bipartition (see e.g. [53]). In short, the neat effect of the interaction is to link the Z -infinite BDI phases, occurring in the absence of it, in eight (Z_8) disconnected (families of) phases, two of them still being (continuously connected with) the ones of the SR Kitaev chain. More in particular, it has been found [58, 59] that the SR one-dimensional phases in the BDI class display, in the absence of interactions between the Kitaev chains forming the ladder, a minimal 2^n degeneracy for the the Schmidt multiplets. This degeneracy is partly lifted by an interaction (even only perturbative), connecting the free phases and leaving the eight distinct phases mentioned above. In particular two of these eight phases display the same ES of the two phases of the SR Kitaev chain.

Appendix B: Long-range paired Kitaev chain

In [27, 35, 36] a quadratic quantum model involving spinless fermions on a one-dimensional lattice have been studied extensively. This is characterized by a long-range (LR) pairing:

$$H_{\text{lat}} = -w \sum_{j=1}^L \left(a_j^\dagger a_{j+1} + \text{h.c.} \right) - \mu \sum_{j=1}^L \left(n_j - \frac{1}{2} \right) + \frac{\Delta}{2} \sum_{j=1}^L \sum_{\ell=1}^{L-1} d_\ell^{-\alpha} \left(a_j a_{j+\ell} + a_{j+\ell}^\dagger a_j^\dagger \right). \quad (\text{B1})$$

For a closed chain, we define in Eq. (B1) $d_\ell = \ell$ ($d_\ell = L - \ell$) if $\ell < L/2$ ($\ell > L/2$) and we choose anti-periodic boundary conditions [27].

The spectrum of excitations is obtained via a Bogoliubov transformation and it is given by ($\omega = \frac{\Delta}{2} \equiv 1$):

$$\lambda_\alpha(k_n) = \sqrt{(\mu - \cos k_n)^2 + f_\alpha^2(k_n + \pi)}. \quad (\text{B2})$$

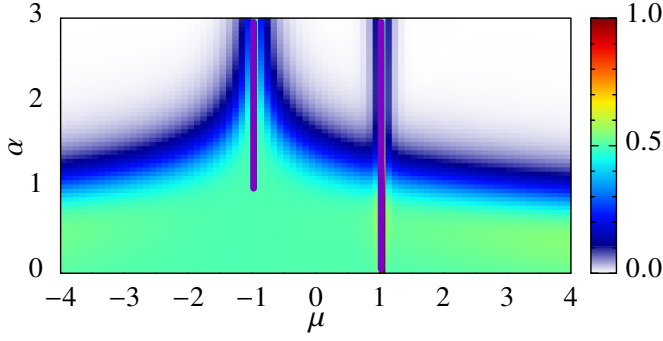


FIG. 9. Phase diagram of the LR paired Kitaev chain in Eqs. (1) and (B1), derived analyzing the area-law deviation for the Von Neumann entropy. We report in particular the quantity c_{eff} defined in Eq. (6). The purple (semi-)lines are critical, there the mass gap vanishes, moreover $c_{\text{eff}} = \frac{1}{2}$ at $\alpha > 1$ and $c_{\text{eff}} = 1$ at $\alpha < 1$ [35].

In Eq. (B2), $k_n = -\pi + 2\pi(n + 1/2)/L$, with $0 \leq n < L$ and $f_\alpha(k) \equiv \sum_{l=1}^{L-1} \sin(kl)/d_l^\alpha$. For sake of simplicity, in the following the subscript n will be neglected. The functions $f_\alpha(k)$ can be also evaluated in the thermodynamic limit, where they become polylogarithmic functions [93]. The ground-state of Eq. (B1) is given by $|\text{GS}\rangle = \prod_{n=0}^{L/2-1} (\cos \theta_k - i \sin \theta_k a_k^\dagger a_{-k}^\dagger) |0\rangle$, with $\tan(2\theta_k) = -f_\alpha(k + \pi)/(\mu - \cos k)$; notably it is even under the Z_2 parity symmetry of the fermionic number (see the Appendix A).

Interestingly the present Hamiltonian can be approximately simulated in the particular case $\alpha = 1$ [94]. The stability against finite-size effects of similar LR phases in current experiments, where only limited sizes can be achieved at the present time, is discussed in the main text, Section VIII.

The phase diagram of the Hamiltonian in Eq. (1) is reported in Fig. 9, plotting the area-law violation for the Von Neumann entropy, quantified as described in the main text. The critical (semi-)lines are also drawn.

Two indicators for the LR phases at $\alpha < 1$

We discuss here two parameters able to detect the LR phases at $\alpha < 1$ of the LR paired Kitaev Hamiltonian in Eq. (B1).

a) connected with the superconductive nature the Hamiltonian in Eq. (B1), we find that a valuable parameter to detect the LR phases is:

$$\langle r^2 \rangle = \sqrt{-\frac{\int dk (u_k^* v_k^*) \nabla_k^2 (u_k v_k)}{\int dk |u_k v_k|^2}} \quad (\text{B3})$$

measuring the average radius of a Cooper pair [95]. Indeed we see in Fig. 10 that, as α is decreased at fixed μ , $\langle r^2 \rangle$ rapidly increases around $\alpha = 1$. Notably the parameter in Eq. (B3) is expected to be a valuable detector of LR phases in superconductive systems no matter their dimensionality.

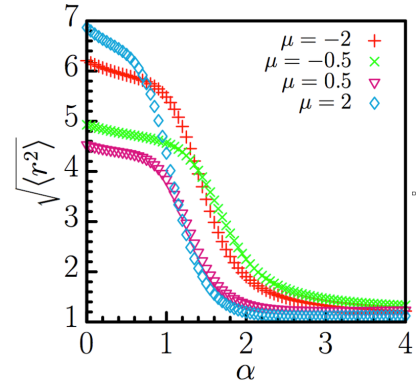


FIG. 10. Average radius of a Cooper pair for the Hamiltonian in Eq. (1), as a function of μ and α .

b) the phase with massive edge states at $\alpha < 1$ and $\mu < 1$ can be also characterized and shown disconnected from the disordered phase above $\alpha = 1$ analyzing the behaviour of the quantity $m_\alpha(\mu)$ [35, 75] defined in the Section VIII. This parameter characterizes when $\alpha \rightarrow \infty$ the paramagnetic-(anti-)ferromagnetic quantum phase transition of the SR Ising model. Indeed in the same limit $m_\alpha(\mu)$ coincides [75] with the modulus of the average longitudinal magnetization

$$|\langle \sigma_x \rangle| = \lim_{l \rightarrow \infty} \sqrt{|\langle \sigma_i^{(x)} \sigma_{i+l}^{(x)} \rangle|} \quad (\text{B4})$$

of the SR Ising model: in particular it has non vanishing values when $|\mu| < 1$ only. The same identification holds at finite α , where $\sigma_i^{(x)}$ refer to the nonlocal spin model obtained from the Hamiltonian in Eq. (B1) by Jordan-Wigner transformation (see e.g. [70]):

$$H_\sigma = -w \sum_j \left(\sigma_j^{(+)} \sigma_{j+1}^{(-)} + \text{H.c.} \right) - \frac{\mu}{2} \sum_j \sigma_j^{(z)} - \frac{\Delta}{2} \sum_{j,\ell} \frac{1}{\ell^\alpha} \left[\sigma_j^{(-)} \exp \left(i\pi \sum_{m=1}^{\ell-1} \sigma_{j+m}^{(+)} \sigma_{j+m}^{(-)} \right) \sigma_{j+\ell}^{(-)} + \text{h.c.} \right], \quad (\text{B5})$$

where $\sigma_j^{(\pm)} = \sigma_j^{(x)} \pm i \sigma_j^{(y)}$.

We plot in Fig. 11 the quantity $m_\alpha(\mu)$ for $L = 800$, $l = \frac{L}{2} = 400$ and different values of μ and α . There $m_\alpha(\mu)$ appears to be vanishing approximately above $\alpha = 1$ (finite-size scaling indicates that this threshold is exactly saturated in the limit $L \rightarrow \infty$) if $|\mu| > 1$, as required for the paramagnetic (disordered) phase proper of the SR limit. On the contrary, at $\alpha < 1$ it assumes a nonzero value when also $\mu < 1$, signaling the outcome of the LR phase with massive edge modes. Notably the vanishing of $m_\alpha(\mu)$ when $\alpha < 1$ and $\mu > 1$, suggests that the nonzero result when $\mu < 1$ cannot be originated from finite-size effects, here made important by the LR nature of the considered model.

It is also remarkable that, even at finite L , the zone where $m_\alpha(\mu) \neq 0$ coincides within very good approximation with

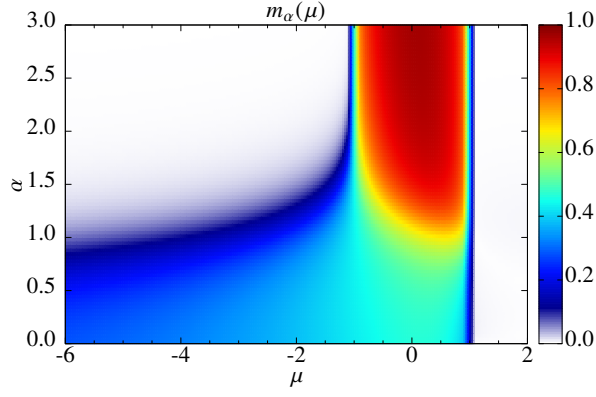


FIG. 11. Plot of the quantity $m_\alpha(\mu)$ in Eq. (B4) for the LR paired Kitaev chain in Eqs. (1) and (B1). We assumed $L = 800$ and different values of μ and α . Notice that if $\mu > 1$, $m_\alpha(\mu)$ is found vanishing for every α .

the zone (at $\mu < 1$) where the violation of the area-law for the Von Neumann entropy takes place (see [27] for a comparison).

We comment finally that Eq. (A2) implies that for the nonlocal spin model in Eq. (B5) the Z_2 spin-flip symmetry is exactly realized in the regime $\alpha < 1$ and $\mu < 1$ and therefore the expectation value of $\sigma_i^{(x)}$ would be expected to vanish: $\langle \sigma_i^{(x)} \rangle = 0$. For this, reason the opposite nonvanishing of $m_\alpha(\mu)$ in the same regime, shown in Fig. 11, suggests a violation of the cluster decomposition property ($\langle \sigma_i^{(x)} \sigma_{i+l}^{(x)} \rangle \rightarrow \langle \sigma_i^{(x)} \rangle \langle \sigma_{i+l}^{(x)} \rangle$ as $l \rightarrow \infty$) for the model in Eq. (B5). Due to the nonlocal nature of the Jordan-Wigner transformation, originating this effect, the same statement cannot be concluded instead for the fermionic Hamiltonian in Eq. (B1). For other examples of nonlocality induced by the same transformation see [82].

Appendix C: Failure of σ -model construction

In this Appendix we show the failure, in the presence of singularities in $H(\mathbf{k})$, of the nonlinear σ -model construction leading to the TWC, at least as derived following the standard approach [63–68]. Here we briefly sketch that derivation without entering too much into the detail and referring to the cited literature for technicalities.

The starting point is the observation that the metal or the insulating nature of a disordered system is usually described by the behavior of the disorder averaging of the diffusion propagator [64]

$$\langle G_{E+w/2}^R(\mathbf{r}, \mathbf{r}') G_{E+w/2}^A(\mathbf{r}, \mathbf{r}') \rangle_{\text{disorder}}, \quad (\text{C1})$$

where $G_E^{R,A}(\mathbf{r}, \mathbf{r}') = \langle \mathbf{r} | (E - H \pm i\eta) | \mathbf{r}' \rangle$ are the retarded/advanced single particle Green's functions, $\eta \rightarrow 0$ is a real infinitesimal value implementing the usual Feynman prescription (see e.g. [96]). The total Hamiltonian H contains

the free part, here denoted as H_0 , and a disorder term

$$H_D = \sum_i \epsilon_i a_i^\dagger a_i. \quad (\text{C2})$$

The random variable ϵ_i is supposed to be normally distributed:

$$P(\epsilon_i) \propto e^{-\epsilon_i^2/4v}. \quad (\text{C3})$$

Introducing the grassmann variables ψ and $\bar{\psi}$, one can write in the Euclidean space

$$\frac{1}{E - H \pm i\eta} \propto \int D\bar{\psi} D\psi \psi \bar{\psi} e^{-S},$$

where $S_0 = \int \bar{\psi}(E - H \pm i\eta)\psi$ is disorder dependent.

In order to evaluate the effect of disorder on a certain observable one should make a stochastic averaging of the quantum expectation values of this observable evaluated at different disorder configurations. This procedure is computational demanding. To overcome this problem one can resort to the so-called replica method [64], which allows to perform disorder averaging in terms of quantum expectation values weighted by a replicated Hamiltonian supplemented by a quartic (interaction) term and taking the zero replica limit.

More specifically, the disorder average of the expectation value $\langle \mathcal{O} \rangle$ of a generic operator \mathcal{O} is given by

$$\overline{\langle \mathcal{O} \rangle} = \overline{\text{Tr}(\rho \mathcal{O})} / \overline{Z}, \quad (\text{C4})$$

where $Z = \text{Tr}(\rho)$ is the partition function and ρ the density operator which defines the quantum state.

Since the random variables are present both in the numerator and in the denominator the stochastic averaging is unpractical. However the great advantage of the replica method is that it makes possible to describe the average over disorder of the ratio in the form of the ratio of the averages. Indeed, introducing n independent replicas of the system, we can formally write

$$\overline{\langle \mathcal{O} \rangle} = \lim_{n \rightarrow 0} \overline{\text{Tr} \left(\prod_{\alpha=1}^n \rho_\alpha \mathcal{O}_1 \right)} / \overline{Z^n}, \quad (\text{C5})$$

where \mathcal{O}_1 means that \mathcal{O} acts only on one replicated system. The price to pay is that the effective action acquires a interacting term among the replicas:

$$\begin{aligned} \overline{Z^n} &= \int D\bar{\psi} D\psi \int d\epsilon_i P(\epsilon_i) \exp \left[-S_0 - \sum_\alpha \sum_i \epsilon_i \bar{\psi}_{i\alpha} \psi_{i\alpha} \right] \\ &= \int D\bar{\psi} D\psi \exp \left[-S_0 + v \sum_{\alpha, \beta} \sum_{i, j} \bar{\psi}_{i\alpha} \bar{\psi}_{j\beta} \psi_{j\beta} \psi_{i\alpha} \right], \end{aligned} \quad (\text{C6})$$

where the sum run over the sites (i, j) and the replica (α, β) indices. By means of the Hubbard-Stratonovich transformation, we can decouple this so-obtained quartic term, introducing an

auxiliary matrix field Q . We get therefore an effective action, reading in momentum space as:

$$S = \frac{1}{V} \frac{1}{w} \text{Tr}[Q_{\mathbf{k}}^2] + \sum_{\mathbf{k}, \mathbf{q}} \bar{\Psi}_{\mathbf{k}} \{ [i\eta s_z + E - H_0(\mathbf{k})] \delta_{\mathbf{q},0} + iV^{-1} Q_{-\mathbf{q}} \} \Psi_{\mathbf{k}+\mathbf{q}}. \quad (\text{C7})$$

The symbol $\Psi_{\mathbf{k}}$ denotes a multi-spinor in the replica space $\Psi_{\mathbf{k}} = (\Psi_{\mathbf{k}1}, \Psi_{\mathbf{k}2}, \dots, \Psi_{\mathbf{k}n})$, and in the particle/hole and retarded/advanced (\pm) spaces, so that, explicitly, $\bar{\Psi}_{\mathbf{k}\alpha} = (-\psi_{\mathbf{k}\uparrow+}, \bar{\psi}_{\mathbf{k}\downarrow+}, -\psi_{\mathbf{k}\uparrow-}, \bar{\psi}_{\mathbf{k}\downarrow-})_{\alpha}$, $\Psi_{\mathbf{k}\alpha} = (\bar{\psi}_{\mathbf{k}\uparrow+}, \psi_{\mathbf{k}\downarrow+}, \bar{\psi}_{\mathbf{k}\uparrow-}, \psi_{\mathbf{k}\downarrow-})_{\alpha}^t$, while s_z is the Pauli matrix in the latter space, w a influential constant, and V the volume of the space. Let us call \mathcal{G} the fermionic propagator appearing in Eq. (C7).

Integrating now over the fermionic fields $\Psi_{\mathbf{k}}$ and $\bar{\Psi}_{\mathbf{k}}$, one gets an action which depends only on $Q_{\mathbf{k}}$:

$$S[Q] = \frac{1}{V} \frac{1}{w} \text{Tr}[Q_{\mathbf{k}}^2] - \frac{1}{2} \text{Tr} \ln \mathcal{G}^{-1}. \quad (\text{C8})$$

where Tr is the trace over all the spaces. After finding the saddle point solution Q_{sp} , the quantum fluctuations are such that $Q_{\mathbf{r}}^2 = Q_{sp}^2$, and the action can be written as follows:

$$S[Q] = S[Q_{sp}] - \frac{1}{4} \text{Tr} \ln(1 + G_0 W), \quad (\text{C9})$$

where $G_0^{-1} = (E - H_0)^2 + Q_{sp}^2$, and

$$W = i[Q, H_0] = -\mathbf{J} \cdot \nabla Q, \quad (\text{C10})$$

where $\mathbf{J}(\mathbf{k}) = \nabla_{\mathbf{k}} H_0(\mathbf{k})$ is a current vertex operator. In the consequent gradient expansion we also obtain:

$$\text{Tr}(G_0 W G_0 W) \simeq \text{Tr}(\mathbf{J} G_0 \mathbf{J} G_0) \text{Tr}(\nabla Q \nabla Q), \quad (\text{C11})$$

the desired σ -model.

The expansion in Eq. (C11) fails if $H_0(\mathbf{k})$ diverges (and it is not regularizable without discontinuities, as for the second-type singularities). In this condition the charge $\int d\mathbf{k} \mathbf{J}(\mathbf{k})$ also diverges. For this reason the σ -model characterizing the coset F cannot be constructed. The present discussion leaves open the possibility of the failure of the σ -model construction also when $\mathbf{J}(\mathbf{k})$ only diverges.

Appendix D: Structure of the edge states at $\alpha < 1$

In Section VII C we mentioned for the LR Kitav chains in the Eqs (1) and (3) the impossibility to identify, in the LR regimes at $\alpha < 1$, low-energy states localized separately on the left-hand and the right-hand edges. This impossibility, directly encoded on the ES structure analyzed in Sections IV and VII, has been claimed to be in a one-to-one correspondence with the nonvanishing edge masses found in the same regimes.

In order to substantiate our thesis, it is useful to start from the construction of the Bogoliubov states for quadratic

fermionic Hamiltonians. As it happens in the ordered phase for the open SR Kitaev chain (and for the Hamiltonians in Eqs. (1) and (3)), the fermionic state $|m\rangle$, whose wavefunction is localized symmetrically at both the edges of the chain, can be written as [75]

$$|m\rangle = \eta_m^{\dagger} |GS\rangle = \sum_{i=1}^L (g_{mi} a_i + h_{mi} a_i^{\dagger}) |GS\rangle, \quad (\text{D1})$$

a similar ansatz holding for the other (bulk) eigenstates of the Hamiltonians. Notice that, compared to $|GS\rangle$, $|m\rangle$ differs in the fermionic number/parity by a unit; this fact is encoded in the different sign on the two states of the topological pfaffian invariant discussed in [49].

As suggested by the linearity of the diagonalizing ansatz for the free Hamiltonians in Eqs. (1) and (3) and following what done in the SR limit (where $m = 0$), one could attempt to decompose the state $|m\rangle$, involving symmetrically both the edges, defining two (right and left) edge operators $f_{R/L}$ as follows (see e. g. [28, 97]):

$$\eta_m = \frac{1}{\sqrt{2}} (f_R + e^{i\phi} f_L) \quad (\text{D2})$$

(the pre-factor $\frac{1}{\sqrt{2}}$ testifying the same weight for the two edges and ϕ being a phase constant to be fixed), depending linearly on a_i and a_i^{\dagger} . If $m = 0$, the so constructed operators $f_{R/L}$ ($\phi = \frac{\pi}{2}$), fulfilling the Majorana condition $f_{R/L}^{\dagger} = e^{i\theta_{R/L}} f_{R/L}$, are related with two wavefunctions localized separately on each edge.

For the Hamiltonians in Eqs. (1) and (3), the situation is very different if $\alpha < 1$. Indeed, since $m \neq 0$, the operators $f_{R/L}$ do not fulfill any longer the Majorana condition, as argued in [28]; for this reason the canonical anti-commutation rules for η_m , $\{\eta_m, \eta_m^{\dagger}\} = 1$ imply:

$$\{f_R, f_R^{\dagger}\} = \{f_L, f_L^{\dagger}\} = 1. \quad (\text{D3})$$

In this way, $f_{R/L}$ are usual fermionic operators, able to induce states (as $|R/L\rangle = f_{R/L}^{\dagger} |GS\rangle$) of the Hilbert space for the considered Hamiltonians. The same possibility does not hold instead if $m = 0$, since $\{f_{R/L}, f_{R/L}^{\dagger}\} = \{f_{R/L}, f_{R/L}\} = 0$ and physical states can be constructed only by combinations of them, as in Eq. (D2) (see for instance [97]). However, states as $|R/L\rangle$ do not belong to the Hilbert space of the Hamiltonians in Eqs. (1) and (3), suggesting that the construction in Eq. (D2), although formally possible, is not correct in the absence of the Majorana condition for $f_{R/L}$, then if $m \neq 0$. On the contrary, only the state $|m\rangle$, involving both of the edges, makes sense in this condition.

The other possibility $|m\rangle = f_R^{\dagger} f_L^{\dagger} |GS\rangle$ is ruled out by the linearity of the diagonalization problem for the considered quadratic Hamiltonians, as well as by the fact that the canonical anti-commutation rules $\{a_i, a_j^{\dagger}\} = \delta_{ij}$ allow cancellations of a_i and a_i^{\dagger} only pairwise, then a linear ansatz as in Eq. (D1) cannot be obtained from the quadratic ansatz for $|m\rangle$ just above. Finally, nonlocal ansatzs are discarded since

the beginning, in such a way as not to change the locality property of the excitations in the bulk/edge spectrum (as done by the Jordan-Wigner transformation for the Majorana modes in the SR limit, see e. g. [82]).

We comment finally that, in the light of the discussion performed in the present Appendix, the exciting hypothesis of edge mode fractionalization without interaction (a central ingredient in the presence of genuine topological order, where fractionalization occurs), suggested in [37] on the basis of the fractional windings for $\tilde{H}(k)$ below $\alpha = 1$ (see Section V), seems ruled out.

Appendix E: Fermionic mapping of the long-range Ising model

The LR Ising Hamiltonian

$$H_{\text{LRI}} = \sin \theta \sum_{i=1}^{L-1} \sum_{\ell=1}^{L-i} \frac{1}{\ell^\alpha} \sigma_i^{(x)} \sigma_{i+\ell}^{(x)} + \cos \theta \sum_{i=1}^L \sigma_i^{(z)} \quad (\text{E1})$$

in the main text can be mapped via a Jordan-Wigner transformation to the interacting LR fermionic chain:

$$H_{\text{LRI}} = \sin \theta \sum_{i=1}^{L-1} \sum_{\ell=1}^{L-i} \frac{1}{\ell^\alpha} (a_i^\dagger - a_i) \text{Exp} \left[i\pi \sum_{m=i+1}^{i+\ell-1} n_m \right] \times \\ \times (a_{i+\ell}^\dagger + a_{i+\ell}) + \cos \theta \sum_{i=1}^L (a_i^\dagger a_i - 1/2), \quad (\text{E2})$$

(where a_i are again fermionic operators and $n_m = a_m^\dagger a_m$) still symmetric under the Z_2 fermionic parity. We notice the effects of the string operators $\text{Exp} [i\pi \sum_{m<i} n_m]$, not canceling each others (oppositely to the SR limit) and making the Hamiltonian in Eq. (E2) LR interacting.

Since the Jordan-Wigner transformation is nonlocal, the knowledge of the properties (entanglement content, edge properties) of the Hamiltonian in Eq. (23) does not help to shed light on the same properties of Eq. (E2) (see e. g. [82]). Conversely, the energy spectra and the phase contents of the two models with open boundary conditions are in a one-to-one correspondence (see Section VIII). This fact also indicates the emergence of a LR phase (again separated by a quantum phase transition without mass gap closure and with a unique Z_2 -symmetric ground-state) also in the interacting LR fermionic chain in Eq. (E2). Moreover, as shown in [28], the LR phase of the same fermionic Hamiltonian at $\alpha < 1$ and in anti-ferromagnetic regime is characterized by edge localization of the lowest eigenstates with nonzero energy, spread instead in the bulk at higher values of α [28]. Therefore, again massive edge states appear in this condition, similar to the ones for the LR Kitaev chains. This facts parallels and confirms the existence of a new phase for the LR Ising model.